

# LOCAL HÖLDER REGULARITY FOR SET-INDEXED PROCESSES

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**ABSTRACT.** In this paper, we study the Hölder regularity of set-indexed stochastic processes defined in the framework of Ivanoff-Merzbach. The first key result is a Hölder-continuity Theorem derived from the approximation of the indexing collection by a nested sequence of finite subcollections. Hölder-continuity based on the increment definition for set-indexed processes is also considered. Then, the localization of these properties leads to various definitions of Hölder exponents. Moreover, a pointwise continuity exponent is defined in relation with the weak continuity property for set-indexed processes which only considers single point jumps. In the case of Gaussian processes, almost sure values are proved for the Hölder exponents. As an application, the local regularity of the set-indexed fractional Brownian motion and the Ornstein-Uhlenbeck process are proved to be constant, with probability one.

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## 1. INTRODUCTION

Sample paths properties of stochastic processes have been deeply studied since the 1970s (we refer to Berman [14, 15], Dudley [18, 19], Orey and Pruitt [34], Orey and Taylor [35] and Strassen [36], for the early study of Gaussian paths). Among the large literature dealing with fine analysis of regularity, Hölder exponents continue to be widely used as a local measure of oscillations (see [11, 12, 30, 31, 32, 37] for examples of recent works in this area).

In the two different definitions of pointwise and local Hölder exponents for a stochastic process  $\{X_t; t \in \mathbf{R}_+\}$ , the increment  $X_t - X_s$  is compared with a power  $|t - s|^\alpha$  or  $\rho^\alpha$  inside a ball  $B(t_0, \rho)$  when  $\rho \rightarrow 0$ . As an example, with probability one, the local regularity of fractional Brownian motion  $\{B^H; t \in \mathbf{R}_+\}$  is constant along the path: the pointwise and local Hölder exponents at any  $t \in \mathbf{R}_+$  are equal to the self-similarity index  $H \in (0, 1)$  (e.g. see [22]).

This field of research is also very active in the multiparameter context and a non-exhaustive list of authors and recent works in this area includes Ayache [8], Dalang [16], Khoshnevisan [16, 29], Lévy-Véhel [22], Xiao [33, 38, 39]. Regularity of set-indexed processes is a more complex issue than regularity of processes indexed by  $\mathbf{R}^N$ . The simple continuity property is closely related to the nature of the indexing collection. As an example, Brownian motion indexed by the lower layers of  $[0, 1]^2$  (i.e. the subsets  $A \subseteq [0, 1]^2$  such that  $[0, t] \subseteq A$  for all  $t \in A$ ) is discontinuous with probability one (we refer to [2] or [27] for the detailed proof). From Dudley's Theorem [19], under assumptions on the indexing collection  $\mathcal{A}$  endowed with some metric  $d_{\mathcal{A}}$ ,

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*total boundedness and integrability condition on the entropy function*, any centered Gaussian  $\mathcal{A}$ -indexed process admits a continuous modification (see [5] or [28] for a complete survey and also [6, 7] for more accurate continuity moduli). In particular, if  $\mathcal{A}$  is a Vapnik-Červonenkis class then the  $\mathcal{A}$ -indexed Brownian motion is continuous with probability one.

In the formal set-indexed setting introduced by Ivanoff and Merzbach in the context of set-indexed martingales (see [27]), instead of conditions on the entropy function of the *indexing collection*, Section 2 of the present paper uses properties of lattices to derive a Kolmogorov-like criterion for Hölder-continuity of a set-indexed process. The collection of sets  $\mathcal{A}$  in the measure space  $(\mathcal{T}, m)$  is endowed with a metric  $d_{\mathcal{A}}$  and a nested sequence  $\underline{\mathcal{A}} = (\mathcal{A}_n)_{n \in \mathbb{N}}$  of finite subcollections of  $\mathcal{A}$  such that each element of  $\mathcal{A}$  can be approximated as the decreasing limit (for the inclusion) of its projections on the  $\mathcal{A}_n$ 's. Under assumptions on  $\underline{\mathcal{A}}$  and  $d_{\mathcal{A}}$  which particularly impose that the distance from any  $U \in \mathcal{A}$  to  $\mathcal{A}_n$  can be related to the cardinal  $k_n = \#\mathcal{A}_n$ , roughly by  $d_{\mathcal{A}}(U, \mathcal{A}_n) = O(k_{n+1}^{-1/q_{\mathcal{A}}})$  where  $q_{\mathcal{A}}$  is called the *discretization exponent* of  $(\mathcal{A}_n)_{n \in \mathbb{N}}$ , we prove in Theorem 2.9: If  $\{X_U; U \in \mathcal{A}\}$  is a set-indexed process and  $\alpha, \beta, K$  are positive constants such that  $\mathbf{E}[|X_U - X_V|^\alpha] \leq K d_{\mathcal{A}}(U, V)^{q_{\mathcal{A}} + \beta}$  for all  $U, V \in \mathcal{A}$ , then for all  $\gamma \in (0, \beta/\alpha)$ , there exist a random variable  $h^*$  and a constant  $L > 0$  such that almost surely

$$\forall U, V \in \mathcal{A}; \quad d_{\mathcal{A}}(U, V) < h^* \Rightarrow |X_U - X_V| \leq L d_{\mathcal{A}}(U, V)^\gamma.$$

Another definition for Hölder-continuity can be based on the definition of increments for set-indexed processes. Instead of quantities  $X_U - X_V$ , the increments of a set-indexed process  $\{X_U; U \in \mathcal{A}\}$  are defined on the class  $\mathcal{C}$  of sets  $C = U_0 \setminus \bigcup_{1 \leq k \leq n} U_k$  where  $U_0, U_1, \dots, U_n \in \mathcal{A}$  by the *inclusion-exclusion formula*

$$\Delta X_C = X_{U_0} - \sum_{k=1}^n \sum_{j_1 < \dots < j_k} (-1)^{k-1} X_{U_0 \cap U_{j_1} \cap \dots \cap U_{j_k}}.$$

According to this definition, another way to express the Hölder-continuity of  $X$  is  $|\Delta X_C| \leq L m(C)^\gamma$ , for  $C \in \mathcal{C}$ . This question is clarified in Section 2.2.

The purpose of Hölder exponents is the (optimal) localization of the Hölder-continuity concept. Following the previous discussion, the first definition for local and pointwise Hölder exponents is based on the comparison between  $|X_U - X_V|$  and a power  $d_{\mathcal{A}}(U, V)^\alpha$  or  $\rho^\alpha$  in a ball  $B_{d_{\mathcal{A}}}(U_0, \rho)$  around  $U_0 \in \mathcal{A}$  when  $\rho \rightarrow 0$ . Another definition compares  $|\Delta X_C|$  for  $C = U \setminus \bigcup_{1 \leq k \leq n} V_k$  in  $\mathcal{C}$  with  $d_{\mathcal{A}}(U, U_0) < \rho$  and  $d_{\mathcal{A}}(U_0, V_k) < \rho$  for each  $k$ , to a power  $m(C)^\alpha$  when  $\rho \rightarrow 0$ . These two kind of exponents are precisely defined in Sections 3.1 and 3.2. In Section 4, the different Hölder exponents are linked to the Hölder regularity of projections of the set-indexed process on increasing paths.

The *pointwise continuity* has been introduced in the multiparameter setting in [4] and in the set-indexed setting in [25] as a weak form of continuity. In this definition, the *point mass jumps* are the only kind of discontinuity considered. Without any supplementary condition on the indexing collection, the set-indexed Brownian motion satisfies this property, even on lower layers where it is not continuous. In Section 3.3, we define the pointwise continuity Hölder exponent of a pointwise continuous process  $X$  by a comparison between  $\Delta X_{C_n(t)}$  with a power  $m(C_n(t))^\alpha$  when  $n \rightarrow \infty$ , where  $(C_n(t))_{n \in \mathbb{N}}$  is a decreasing sequence of elements in  $\mathcal{C}$  which converges to  $t \in \mathcal{T}$ .

In the Gaussian case, we prove in Section 5 that the different aforementioned Hölder exponents admit almost sure values. Moreover these almost sure values can be obtained uniformly on  $\mathcal{A}$  under some supplementary conditions. As an application, we consider in Section 6 the case of the set-indexed fractional Brownian motion (SifBm) defined in [23] and the set-indexed Ornstein-Uhlenbeck (SIOU) process defined in [10].

We proved that, with probability one, all the different Hölder exponents (except the pointwise exponent) of the SifBm at any set  $U \in \mathcal{A}$  are equal to  $H$ , the index of self-similarity of the process (extending and improving a result in the multiparameter case which was first proved in [1] and [22]). For SIOU process, they are almost surely equal to  $1/2$  at any set  $U \in \mathcal{A}$ .

## 2. HÖLDER CONTINUITY OF A SET-INDEXED PROCESS

In the classical case of one-parameter (or multiparameter) stochastic processes, Kolmogorov's criterion is a useful tool to study sample paths continuity (e.g. see [19, 28, 22]). In this section, we extend this result to a particular set-indexed setting.

**2.1. Indexing collection for set-indexed processes.** We recall briefly the framework of set-indexed processes (see [27], [23]). Let  $\mathcal{T}$  be a locally compact complete separable metric and measure space with metric  $d$  and Radon measure  $m$  defined on the Borel sets of  $\mathcal{T}$ . All stochastic processes will be indexed by a class  $\mathcal{A}$  of compact connected subsets of  $\mathcal{T}$ .

In the whole paper, the class of finite unions of sets in any collection  $\mathcal{D}$  will be denoted by  $\mathcal{D}(u)$ . In the terminology of [27], we assume that  $\mathcal{A}$  is an *indexing collection*:

**Definition 2.1.** *A nonempty class  $\mathcal{A}$  of compact, connected subsets of  $\mathcal{T}$  is called an indexing collection if it satisfies the following:*

- (1)  $\emptyset \in \mathcal{A}$ , and  $A^\circ \neq A$  if  $A \neq \emptyset$  or  $\mathcal{T}$ . In addition, there is an increasing sequence  $(B_n)_{n \in \mathbb{N}}$  of sets in  $\mathcal{A}(u)$  such that  $\mathcal{T} = \bigcup_{n=1}^{\infty} B_n^\circ$ .
- (2)  $\mathcal{A}$  is closed under arbitrary intersections and if  $A, B \in \mathcal{A}$  are nonempty, then  $A \cap B$  is nonempty. If  $(A_i)$  is an increasing sequence in  $\mathcal{A}$  then  $\overline{\bigcup_i A_i} \in \mathcal{A}$ .
- (3) The  $\sigma$ -algebra generated by  $\mathcal{A}$ ,  $\sigma(\mathcal{A}) = \mathcal{B}$ , the collection of all Borel sets of  $\mathcal{T}$ .
- (4) Separability from above: There exists an increasing sequence of finite subclasses  $\mathcal{A}_n = \{A_1^n, \dots, A_{k_n}^n\}$  of  $\mathcal{A}$  closed under intersections and satisfying  $\emptyset, B_n \in \mathcal{A}_n(u)$  ( $B_n$  is defined in (1)), and a sequence of functions  $g_n : \mathcal{A} \rightarrow \mathcal{A}_n(u)$  such that
  - (a)  $g_n$  preserves arbitrary intersections and finite unions (i.e.  $g_n(\bigcap_{A \in \mathcal{A}'} A) = \bigcap_{A \in \mathcal{A}'} g_n(A)$  for any  $\mathcal{A}' \subseteq \mathcal{A}$ , and if  $\bigcup_{i=1}^k A_i = \bigcup_{j=1}^m A'_j$ , then  $\bigcup_{i=1}^k g_n(A_i) = \bigcup_{j=1}^m g_n(A'_j)$ ),
  - (b) for each  $A \in \mathcal{A}$ ,  $A \subseteq (g_n(A))^\circ$ ,
  - (c)  $g_n(A) \subseteq g_m(A)$  if  $n \geq m$ ,
  - (d) for each  $A \in \mathcal{A}$ ,  $A = \bigcap_n g_n(A)$ ,
  - (e) if  $A, A' \in \mathcal{A}$  then for every  $n$ ,  $g_n(A) \cap A' \in \mathcal{A}$ , and if  $A' \in \mathcal{A}_n$  then  $g_n(A) \cap A' \in \mathcal{A}_n$ .
  - (f)  $g_n(\emptyset) = \emptyset \forall n$ .
- (5) Every countable intersection of sets in  $\mathcal{A}(u)$  may be expressed as the closure of a countable union of sets in  $\mathcal{A}$ .

(Note: ' $\subset$ ' indicates strict inclusion and ' $\overline{(\cdot)}$ ' and ' $(\cdot)^\circ$ ' denote respectively the closure and the interior of a set.)

Since we aim at proving the continuity of  $\mathcal{A}$ -indexed processes, the indexing collection needs to be endowed with some topology. In our situation, the simplest way to do so is to consider a distance on  $\mathcal{A}$ .

Along this paper, we may sometimes specify the distance on  $\mathcal{A}$  that we are using. We present here two of them of special interest:

- The classical Hausdorff metric  $d_H$  defined on  $\mathcal{K} \setminus \emptyset$ , the nonempty compact subsets of  $\mathcal{T}$ , by

$$\forall U, V \in \mathcal{K} \setminus \emptyset; \quad d_H(U, V) = \inf \{ \epsilon > 0 : U \subseteq V^\epsilon \text{ and } V \subseteq U^\epsilon \},$$

where  $U^\epsilon = \{x \in \mathcal{T} : d(x, U) \leq \epsilon\}$ ;

- and the pseudo-distance  $d_m$  defined by

$$\forall U, V \in \mathcal{A}; \quad d_m(U, V) = m(U \triangle V),$$

where  $m$  is the measure on  $\mathcal{T}$  and  $\triangle$  denotes the symmetric difference of sets.

**Remark 2.2.** *In the case of  $\mathcal{A} = \{[0, t]; t \in \mathbf{R}_+^N\}$ ,  $(s, t) \mapsto d_m([0, s], [0, t])$  induces a distance on  $\mathbf{R}_+^N$ . This distance can be compared to the classical distances of the space  $\mathbf{R}^N$ ,*

$$\begin{aligned} d_1 : (s, t) &\mapsto \|t - s\|_1 = \sum_{i=1}^N |t_i - s_i|, \\ d_2 : (s, t) &\mapsto \|t - s\|_2 = \sum_{i=1}^N (t_i - s_i)^2, \\ d_\infty : (s, t) &\mapsto \|t - s\|_\infty = \max_{1 \leq i \leq N} |t_i - s_i|. \end{aligned}$$

*If  $m$  is the Lebesgue measure of  $\mathbf{R}^N$ , the distance  $d_m$  is equivalent to  $d_1$ ,  $d_2$  and  $d_\infty$  on any compact of  $\mathbf{R}_+^N \setminus \{0\}$ .*

*More precisely, for all  $a \prec b$  in  $\mathbf{R}_+^N \setminus \{0\}$ , there exist two positive constants  $m_{a,b}$  and  $M_{a,b}$  such that*

$$\forall s, t \in [a, b]; \quad m_{a,b} d_1(s, t) \leq m([0, s] \triangle [0, t]) \leq M_{a,b} d_\infty(s, t).$$

*We refer to [21] for a proof of these assertions.*

With a view to studying Hölder regularity of stochastic processes, according to Dudley's work in the specific frame of Gaussian processes ([19]), it is reasonable to assume that the indexing collection is at least totally bounded. Following the conditions of Definition 2.1, some additional assumptions on the collection  $\mathcal{A}$  are required to guarantee that  $(\mathcal{A}, d_{\mathcal{A}})$  is totally bounded (or at least locally totally bounded). For sake of completeness, we recall the definitions of some of these notions.

A metric space  $(\mathcal{T}, d)$  is *totally bounded* if for any  $\epsilon > 0$ ,  $\mathcal{T}$  can be covered by a finite number of balls of radius smaller than  $\epsilon$ . The minimal number of such balls is called the *metric entropy* and is denoted  $N(\mathcal{T}, d, \epsilon)$ .

Before getting to the main assumption on  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  and  $d_{\mathcal{A}}$ , we notice that the sequence  $(k_n)_{n \in \mathbf{N}} = (\#\mathcal{A}_n)_{n \in \mathbf{N}}$  is an increasing sequence that tends to  $\infty$ , as  $n \rightarrow \infty$ . This property comes from the combination of (4)(b) and (4)(d) in the definition of an indexing collection.

**Assumption 1.** Let  $d_{\mathcal{A}}$  be a (pseudo-)distance on  $\mathcal{A}$ . Let us suppose that  $\underline{\mathcal{A}} = (\mathcal{A}_n)_{n \in \mathbb{N}}$  satisfies the following assertions:

- (1)  $g_n$  is  $\mathcal{A}_n$ -valued and  $\inf_{n \in \mathbb{N}} (k_{n+1}/k_n) > 1$ ;
- (2) There exist positive real numbers  $q_{\underline{\mathcal{A}}}$ ,  $M_1$  and  $M_2$  such that, for all  $U \in \mathcal{A}$ ,

$$\inf_{\substack{V \in \mathcal{A}_n \\ U \subset V^\circ}} d_{\mathcal{A}}(U, V) \leq M_1 k_{n+1}^{-1/q_{\underline{\mathcal{A}}}} \quad (\text{H1})$$

and for all  $\rho > 0$ ,

$$\#(B(U, \rho) \cap \mathcal{A}_n) \leq M_2 k_n \rho^{q_{\underline{\mathcal{A}}}}, \quad (\text{H2})$$

where  $B(U, \rho)$  denotes the set of  $W \in \mathcal{A}$  with  $d_{\mathcal{A}}(U, W) < \rho$ .

The real  $q_{\underline{\mathcal{A}}}$  is not unique and it depends a priori on the sub-semilattices  $\underline{\mathcal{A}} = (\mathcal{A}_n)_{n \in \mathbb{N}}$  and the distance  $d_{\mathcal{A}}$ . Such a real  $q_{\underline{\mathcal{A}}}$  is called *discretization exponent* of  $(\mathcal{A}_n)_{n \in \mathbb{N}}$ .

The functions  $(g_n)_{n \in \mathbb{N}}$  in Definition 2.1 can be considered as (outer) projections on the sub-semilattices  $(\mathcal{A}_n)_{n \in \mathbb{N}}$ , and Condition (H1) implies that these projections converge fast enough compared to  $k_n$ . Practically, the set  $g_n(U)$  can be considered to realize the minimum of  $d_{\mathcal{A}}(U, V)$  where  $V \in \mathcal{A}_n$  and  $U \subset V^\circ$ , up to a modification of  $g_n$ 's definition in:

$$\forall U \in \mathcal{A}, \quad g_n(U) := \bigcap_{\substack{V \in \mathcal{A}_n \\ U \subset V^\circ}} V. \quad (2.1)$$

In the sequel, the functions  $g_n$  are always supposed to satisfy this equality. Thus Condition (H1) becomes:

$$\sup_{U \in \mathcal{A}} d_{\mathcal{A}}(U, g_n(U)) \leq M_1 k_{n+1}^{-1/q_{\underline{\mathcal{A}}}}. \quad (2.2)$$

Moreover for any integer  $n$ , the triangular inequality leads to

$$\sup_{U \in \mathcal{A}} d_{\mathcal{A}}(g_{n+1}(U), g_n(U)) \leq 2M_1 k_{n+1}^{-1/q_{\underline{\mathcal{A}}}}. \quad (2.3)$$

This last equality is of crucial importance to prove the existence of a Hölder continuous modification for an  $\mathcal{A}$ -indexed process. It enables chaining arguments in the indexing collection  $\mathcal{A}$ .

The following example shows that Assumption 1 is satisfied in simple situations.

**Example 2.3.** • In the case of  $\mathcal{A} = \{[0, t]; t \in [0, 1] \subset \mathbf{R}_+\}$ , the subclasses  $\mathcal{A}_n$  are commonly  $\{[0, k \cdot 2^{-n}]; k = 0, \dots, 2^n\}$ . The Hausdorff and the symmetric difference distance on  $\mathcal{A}$  are the same,  $d_{\mathcal{A}} : ([0, s], [0, t]) \mapsto |t - s|$ , and we have

$$\forall k = 0, \dots, 2^n - 1; \quad d_{\mathcal{A}}([0, k \cdot 2^{-n}], [0, (k+1) \cdot 2^{-n}]) = 2^{-n}.$$

Then the conditions of Assumption 1 are all trivially satisfied for  $q_{\underline{\mathcal{A}}} = 1$ .

- In the case of  $\mathcal{A} = \{[0, t]; t \in [0, 1] \subset \mathbf{R}_+^N\}$ , the subclasses  $\mathcal{A}_n$  can be

$$\{[0, 2^{-n} \cdot (l_1, \dots, l_N)]; 0 \leq l_1, \dots, l_N \leq 2^n\}.$$

Let  $U$  be a set in  $\mathcal{A}$ . The distance (induced by the Lebesgue measure  $\lambda$ ) between  $U$  and  $g_n(U)$  will be the volume difference between the two sets. It is easily

majorated by the sum of the volumes of the outer faces, minus a residue that is negligible

$$\sup_{U \in \mathcal{A}} d_\lambda(U, g_n(U)) = \sup_{U \in \mathcal{A}} \lambda(g_n(U) \setminus U) = N \cdot 2^{-n} + o(2^{-n}).$$

Since  $k_n = (2^n + 1)^N$ , it yields naturally to

$$d_\lambda(U, g_n(U)) = O(k_{n+1}^{-1/q_A}),$$

with  $q_A = N$  and all the other conditions of Assumption 1 are satisfied.

On the contrary to the rectangles case, the following result shows that the collection of *lower layers* of  $\mathbf{R}^N$  does not satisfy Assumption 1. We will see later that this result is not surprising in the view of Theorem 2.9 since Brownian motion indexed by the lower layers of  $[0, 1]^2$  does not have a continuous modification, as can be seen for instance in [2, 27].

**Lemma 2.4.** *Let  $\mathcal{A}$  be the collection of lower layers of  $[0, 1]^2$ , i.e. the subsets  $A$  of  $[0, 1]^2$  such that  $\forall t \in A, [0, t] \subseteq A$ . For all  $n \in \mathbf{N}$ , let  $\mathcal{A}_n$  be the collection of finite unions of sets in the dissecting collection of the dyadic rectangles of  $[0, 1]^2$ , i.e.*

$$\mathcal{A}_n = \left\{ \bigcup_{\text{finite}} [0, x] : 2^n x \in \mathbf{Z}^2 \cap (0, 2^n]^2 \right\} \cup \{0\} \cup \{\emptyset\}.$$

Then, the cardinal  $k_n$  of  $\mathcal{A}_n$  satisfies  $k_n \geq 2^{2^n}$  for all  $n \in \mathbf{N}$ .

Consequently, Condition (H1) of Assumption 1 does not hold in that case.

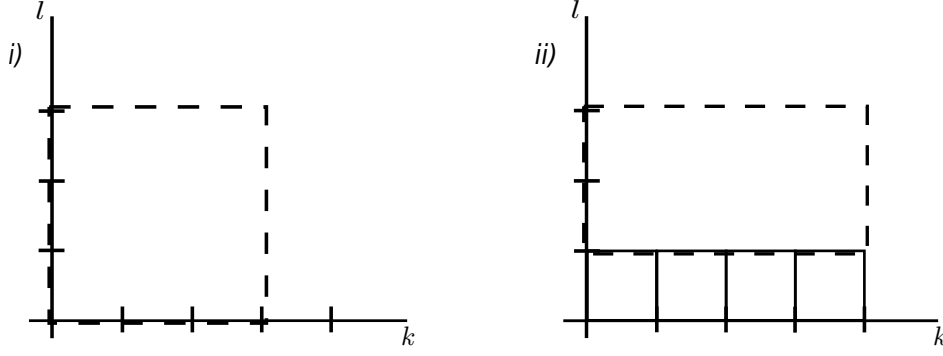
We only give a sketch of the proof of Lemma 2.4, since the result will be improved as a corollary of Theorem 2.9 in the next Section.

*Sketch of proof.* Following (2.2), we need to compare the cardinal  $k_n$  of  $\mathcal{A}_n$  (as  $n$  goes to  $\infty$ ) to:

$$\sup_{U \in \mathcal{A}} d_\lambda(U, g_n(U)) \geq \sup_{\substack{U \in \mathcal{A} \\ U \text{ rectangle}}} d_\lambda(U, g_n(U)) \approx 2^{-n+1}.$$

Let  $n \geq 0$  be a fixed integer. For  $k, l \in \llbracket 1, 2^n \rrbracket$ , let  $d_{k,l}$  be the number of lower layers of  $\mathcal{A}_n$  that are inside the rectangle  $[0, (k \cdot 2^{-n}, l \cdot 2^{-n})]$ . With these notations, we have  $k_n = d_{2^n}$  (denoting  $d_l = d_{l,1}$ ).

Using Figure 1, we build and count the number of lower layers (of  $\mathcal{A}_n$ ) for  $k$  and  $l$  fixed. In a first step (i) in the figure), we enumerate the lower layers inside the dotted region, that is for a region of width  $k-1$  and height  $l$ . This gives  $d_{k-1,l}$  lower layers. In a second time (ii) in the figure), we assume that all the lower layers we are interested in have a full first row. There are precisely  $d_{k,l-1}$  such lower layers.

FIGURE 1. lower layers for  $k = 4, l = 3$ 

In the construction previously described, we never count twice the same lower layer. As a consequence, we can claim

$$\forall k, l \geq 1, \quad d_{k,l} \geq d_{k,l-1} + d_{k-1,l}.$$

Actually, we can be convinced that the equality holds, i.e.  $d_{k,l} = d_{k,l-1} + d_{k-1,l}$  for all  $k, l \geq 1$ .

By symmetry in  $k$  and  $l$ , we notice that for all  $n \in \mathbf{N}$ ,

$$d_{n+1} = d_{n+1,n+1} \geq 2 d_{n,n+1} \geq 2(d_{n,n} + d_{n+1,n-1}) \geq 2 d_n.$$

Since  $d_1 = 2$ , we deduce that  $d_n \geq 2^n$  for all  $n \geq 1$  and, as a consequence,

$$\forall n \geq 1, \quad k_n = d_{2^n} \geq 2^{2^n}.$$

As  $k_n$  is growing faster than  $2^{2^n}$ , there cannot exist any real  $q_{\mathcal{A}} > 0$  such that  $\sup_{U \in \mathcal{A}} d_{\lambda}(U, g_n(U)) = O(k_{n+1}^{-1/q_{\mathcal{A}}})$ . Condition (H1) of Assumption 1 is not satisfied.  $\square$

The fact that  $g_n$  is  $\mathcal{A}_n$ -valued in Assumption 1 is satisfied if  $\mathcal{T} \in \mathcal{A}_n$  for all  $n \in \mathbf{N}$ . In that case, Condition (4)(e) of an indexing collection implies then that  $g_n(U) \cap \mathcal{T} \in \mathcal{A}_n$ , for all  $U \in \mathcal{A}$ .

Another way to satisfy this condition is to assume that the subset  $B_n$  (of point (1) of Definition 2.1), belongs to every subclasses  $\mathcal{A}_n$ . In that case,  $g_n(U) \cap B_n \in \mathcal{A}_n$  for all  $U \in \mathcal{A}$ , and the function  $g_n$  can be  $\mathcal{A}_n$ -valued without restriction, up to substitution with  $\tilde{g}_n$  defined by  $\tilde{g}_n(U) = g_n(U) \cap B_n$ .

To conclude this section, we emphasize the fact that Assumption 1 implies the total boundedness of  $(\mathcal{A}, d_{\mathcal{A}})$ : Since

$$\forall n \in \mathbf{N}, \quad d_{\mathcal{A}}(U, g_n(U)) \leq M_1 k_n^{-1/q_{\mathcal{A}}},$$

$\mathcal{A}_n$  constitutes a  $k_n^{-1/q_{\mathcal{A}}}$ -net for all  $n \in \mathbf{N}$ , and thus  $(\mathcal{A}, d_{\mathcal{A}})$  is totally bounded.

**2.2. Kolmogorov's criterion.** As  $(\mathcal{A}, d_{\mathcal{A}})$  is not generally totally bounded, for any deterministic function  $f : \mathcal{A} \rightarrow \mathbf{R}$ , we consider the *modulus of continuity* on any totally bounded  $\mathcal{B} \subset \mathcal{A}$

$$\omega_{f,\mathcal{B}}(\delta) = \sup_{\substack{U, V \in \mathcal{B} \\ d_{\mathcal{A}}(U, V) \leq \delta}} |f(U) - f(V)|.$$



Recall that the function  $f$  is said *Hölder continuous of order  $\alpha > 0$*  if for all totally bounded  $\mathcal{B} \subset \mathcal{A}$  one of the following equivalent conditions holds (e.g. see [28], Chapter 5)

(i.)

$$\limsup_{\delta \rightarrow 0} \delta^{-\alpha} \omega_{f, \mathcal{B}}(\delta) < \infty.$$

(ii.) There exists  $M > 0$  and  $\delta_0 > 0$  such that for all  $U, V \in \mathcal{B}$  with  $d_{\mathcal{A}}(U, V) < \delta_0$ ,  $|f(U) - f(V)| \leq M.d_{\mathcal{A}}(U, V)^\alpha$ .

For any general set-indexed Gaussian process, Dudley's Corollary 2.3 in [19] allows to compute a sample modulus (giving the same kind of result than Corollary 2.10). But the results holds under certain conditions on the indexing collection, namely the totally boundedness for the considered distance, and some control of the metric entropy. In the case where the indexing collection is a Vapnik-Červonenkis class, the condition on the entropy becomes more strict. Actually, Alexander showed in [7] that an even more accurate sample modulus, of the iterated logarithm type, could be achieved for additive set-indexed Brownian motion.

Although Adler and Taylor emphasize in [5] on the fact that, in the proof of the previous results, the Gaussian property is only used through the exponential Tchebycheff inequality, they do not suggest any Kolmogorov Criterion for non-gaussian processes. The following Theorem 2.9 do so in the general set-indexed framework of Ivanoff and Merzbach, thanks to the discretization exponent.

**Definition 2.5.** A (pseudo-)distance  $d_{\mathcal{A}}$  on  $\mathcal{A}$  is said:

- (i) Outer-continuous if for any non-increasing sequence  $(U_n)_{n \in \mathbf{N}}$  in  $\mathcal{A}$  converging to  $U = \bigcap_{n \in \mathbf{N}} U_n \in \mathcal{A}$ ,  $d_{\mathcal{A}}(U_n, U)$  tends to 0 as  $n$  goes to  $\infty$  ;
- (ii) Contracting if it is outer-continuous and if for any  $U$  and  $V$  in  $\mathcal{A}$ ,

$$d_{\mathcal{A}}(U, U \cap V) \leq d_{\mathcal{A}}(U, V).$$

The outer-continuity property for a distance on  $\mathcal{A}$  allows to give an upper bound for  $d_{\mathcal{A}}(g_n(U), g_m(V))$  when  $n$  and  $m$  go to  $\infty$ .

**Lemma 2.6.** Let  $d_{\mathcal{A}}$  be a distance on  $\mathcal{A}$  satisfying either the outer-continuity property or Assumption 1. Then, for any  $U, V \in \mathcal{A}$ , we have

$$\forall \epsilon > 0, \exists n_0 \in \mathbf{N}, \forall n, p \in \mathbf{N},$$

$$[n \geq n_0, p \geq n_0] \Rightarrow d_{\mathcal{A}}(g_n(U), g_p(V)) \leq d_{\mathcal{A}}(U, V) + \epsilon. \quad (2.4)$$

*Proof.* From outer-continuity of  $d_{\mathcal{A}}$ , one deduce that  $d_{\mathcal{A}}(g_n(U), U)$  converges to 0 as  $n$  goes to  $\infty$ . Then, the triangular inequality implies

$$d_{\mathcal{A}}(g_n(U), g_p(V)) \leq d_{\mathcal{A}}(g_n(U), U) + d_{\mathcal{A}}(U, V) + d_{\mathcal{A}}(V, g_p(V)).$$

The integer  $n_0$  is defined such that for all integer  $n \geq n_0$ ,  $d_{\mathcal{A}}(g_n(U), U) \leq \epsilon/2$  and  $d_{\mathcal{A}}(g_n(V), V) \leq \epsilon/2$ .

The case of  $d_{\mathcal{A}}$  not outer-continuous but satisfies Assumption 1 follows directly from inequality (2.2) and triangular inequality.  $\square$

The result of Lemma 2.6 can be improved in the specific case of the metric induced by the measure  $m$ .



**Lemma 2.7.** *For any  $U, V \in \mathcal{A}$ , we have*

$$\lim_{n, p \rightarrow \infty} m(g_n(U) \triangle g_p(V)) = m(U \triangle V). \quad (2.5)$$

*Proof.* Fix  $n \in \mathbf{N}$  and let  $p \rightarrow \infty$ .

We can write

$$m(g_n(U) \triangle g_p(V)) = m(g_n(U) \setminus g_p(V)) + m(g_p(V) \setminus g_n(U)).$$

As for all  $n \in \mathbf{N}$ ,

$$\begin{cases} \forall p \in \mathbf{N}; & [g_n(U) \setminus g_p(V)] \subset [g_n(U) \setminus g_{p+1}(V)] \\ \bigcup_{p \in \mathbf{N}} [g_n(U) \setminus g_p(V)] = [g_n(U) \setminus V], \end{cases}$$

we have

$$\forall n \in \mathbf{N}; \quad \lim_{p \rightarrow \infty} m(g_n(U) \setminus g_p(V)) = m(g_n(U) \setminus V). \quad (2.6)$$

In the same way, as for all  $n \in \mathbf{N}$ ,

$$\begin{cases} \forall p \in \mathbf{N}; & [g_{p+1}(V) \setminus g_n(U)] \subset [g_p(V) \setminus g_n(U)] \\ \bigcap_{p \in \mathbf{N}} [g_p(V) \setminus g_n(U)] = [V \setminus g_n(U)], \end{cases}$$

we have

$$\forall n \in \mathbf{N}; \quad \lim_{p \rightarrow \infty} m(g_p(V) \setminus g_n(U)) = m(V \setminus g_n(U)). \quad (2.7)$$

From (2.6) and (2.7), we get

$$\forall n \in \mathbf{N}; \quad \lim_{p \rightarrow \infty} m(g_n(U) \triangle g_p(V)) = m(g_n(U) \triangle V). \quad (2.8)$$

In the same way, we show that

$$\lim_{n \rightarrow \infty} m(g_n(U) \triangle V) = m(U \triangle V). \quad (2.9)$$

Then, expression (2.5) follows from (2.8) and (2.9).  $\square$

We can prove easily that equality (2.5) also holds for  $d_H$ . The following result is also stated without any proof, which is direct.

**Proposition 2.8.** *The metrics  $d_m$  and  $d_H$  are contracting.*

Assumption 1 on the subcollections  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  and the metric  $d_{\mathcal{A}}$  allows to state the following result:

**Theorem 2.9.** *Let  $d_{\mathcal{A}}$  be a (pseudo-)distance on the indexing collection  $\mathcal{A}$ , whose subclasses  $\underline{\mathcal{A}} = (\mathcal{A}_n)_{n \in \mathbf{N}}$  satisfy Assumption 1 with a discretization exponent  $q_{\underline{\mathcal{A}}} > 0$ . Let  $X = \{X_U; U \in \mathcal{A}\}$  be a set-indexed process such that*

$$\forall U, V \in \mathcal{A}; \quad \mathbf{E}[|X_U - X_V|^\alpha] \leq K d_{\mathcal{A}}(U, V)^{q_{\underline{\mathcal{A}}} + \beta} \quad (2.10)$$

where  $K$ ,  $\alpha$  and  $\beta$  are positive constants.

Then, the sample paths of  $X$  are almost surely locally  $\gamma$ -Hölder continuous for all  $\gamma \in (0, \frac{\beta}{\alpha})$ , i.e. there exist a random variable  $h^*$  and a constant  $L > 0$  such that almost surely

$$\forall U, V \in \mathcal{A}; \quad d_{\mathcal{A}}(U, V) < h^* \Rightarrow |X_U - X_V| \leq L d_{\mathcal{A}}(U, V)^\gamma.$$

*Proof.* Let us fix  $\gamma \in (0, \frac{\beta}{\alpha})$ . For all  $U, V \in \mathcal{A}_n$  such that  $d_{\mathcal{A}}(U, V) \leq 3M_1.k_n^{-1/q_{\mathcal{A}}}$ , we have

$$\begin{aligned} \mathbf{P}(|X_U - X_V| \geq d_{\mathcal{A}}(U, V)^\gamma) &\leq \frac{\mathbf{E}[|X_U - X_V|^\alpha]}{d_{\mathcal{A}}(U, V)^{\alpha\gamma}} \leq K d_{\mathcal{A}}(U, V)^{q_{\mathcal{A}}+\beta-\alpha\gamma} \\ &\leq K (3M_1)^{q_{\mathcal{A}}+\beta-\alpha\gamma} k_n^{-1-(\beta-\alpha\gamma)/q_{\mathcal{A}}}. \end{aligned}$$

Then using Condition (H2) of Assumption 1,

$$\begin{aligned} \mathbf{P}\left(\max_{\substack{U, V \in \mathcal{A}_n \\ d_{\mathcal{A}}(U, V) \leq 3M_1.k_n^{-1/q_{\mathcal{A}}}}} \frac{|X_U - X_V|}{d_{\mathcal{A}}(U, V)^\gamma} \geq 1\right) &\leq \sum_{\substack{U, V \in \mathcal{A}_n \\ d_{\mathcal{A}}(U, V) \leq 3M_1.k_n^{-1/q_{\mathcal{A}}}}} \mathbf{P}(|X_U - X_V| \geq d_{\mathcal{A}}(U, V)^\gamma) \\ &\leq K (3M_1)^{q_{\mathcal{A}}+\beta-\alpha\gamma} M_2 k_n^2 (3M_1.k_n^{-1/q_{\mathcal{A}}})^{q_{\mathcal{A}}} k_n^{-1-(\beta-\alpha\gamma)/q_{\mathcal{A}}} \\ &\leq K (3M_1)^{2q_{\mathcal{A}}+\beta-\alpha\gamma} M_2 k_n^{-(\beta-\alpha\gamma)/q_{\mathcal{A}}}. \end{aligned}$$

From Assumption 1, there exists a real  $a > 1$  such that  $k_{n+1} \geq a.k_n$  for all  $n \geq 1$ . Then, since  $\beta - \alpha\gamma > 0$ , Borel-Cantelli's theorem implies the existence of  $\Omega^* \subset \Omega$  with  $\mathbf{P}(\Omega^*) = 1$  such that  $\forall \omega \in \Omega^*$ ,

$$\begin{aligned} \exists n^*(\omega) \in \mathbf{N}, \forall n \geq n^*, \forall U, V \in \mathcal{A}_n; \\ d_{\mathcal{A}}(U, V) \leq 3M_1.k_n^{-1/q_{\mathcal{A}}} \Rightarrow |X_U - X_V| < d_{\mathcal{A}}(U, V)^\gamma. \end{aligned} \quad (2.11)$$

Let us consider  $\mathcal{D} = \bigcup_{n \in \mathbf{N}} \mathcal{A}_n$  and recall some properties of the functions  $g_n$ . For any  $U \in \mathcal{A}$ ,  $g_n(U) \downarrow U$  and  $U = \bigcap_{n \in \mathbf{N}} g_n(U)$ . The collections  $\mathcal{A}_n$  will play the role of the projection sets in the entropy arguments, and the  $g_n$ 's the projections themselves. For  $\omega \in \Omega^*$ , consider any  $U, V \in \mathcal{D}$  such that  $d_{\mathcal{A}}(U, V) < M_1 k_n^{-1/q_{\mathcal{A}}}$  for some  $n \geq n^*$ . We can write

$$X_U = X_{g_n(U)} + \sum_{l \geq n} (X_{g_{l+1}(U)} - X_{g_l(U)}) \quad \text{and} \quad X_V = X_{g_n(V)} + \sum_{l \geq n} (X_{g_{l+1}(V)} - X_{g_l(V)}).$$

Then,

$$|X_U - X_V| \leq |X_{g_n(U)} - X_{g_n(V)}| + \sum_{l \geq n} |X_{g_{l+1}(U)} - X_{g_l(U)}| + \sum_{l \geq n} |X_{g_{l+1}(V)} - X_{g_l(V)}|. \quad (2.12)$$

Using the inequality

$$\begin{aligned} d_{\mathcal{A}}(g_n(U), g_n(V)) &\leq d_{\mathcal{A}}(U, g_n(U)) + d_{\mathcal{A}}(U, V) + d_{\mathcal{A}}(V, g_n(V)) \\ &\leq 2M_1 k_{n+1}^{-1/q_{\mathcal{A}}} + d_{\mathcal{A}}(U, V) \leq 3M_1 k_n^{-1/q_{\mathcal{A}}}, \end{aligned}$$

we deduce from (2.11)

$$|X_{g_n(U)} - X_{g_n(V)}| < d_{\mathcal{A}}(g_n(U), g_n(V))^\gamma \leq \left(2M_1 k_{n+1}^{-1/q_{\mathcal{A}}} + d_{\mathcal{A}}(U, V)\right)^\gamma. \quad (2.13)$$

For each  $l \geq n$ , the inequalities  $d_{\mathcal{A}}(g_{l+1}(U), g_l(U)) \leq 2M_1 k_{l+1}^{-1/q_{\mathcal{A}}}$  (see (2.3)) and  $d_{\mathcal{A}}(g_{l+1}(V), g_l(V)) \leq 2M_1 k_{l+1}^{-1/q_{\mathcal{A}}}$  imply by (2.11)

$$\begin{aligned} \sum_{l \geq n} |X_{g_{l+1}(U)} - X_{g_l(U)}| + \sum_{l \geq n} |X_{g_{l+1}(V)} - X_{g_l(V)}| &\leq 2(2M_1)^\gamma \sum_{l \geq n} k_{l+1}^{-\gamma/q_{\mathcal{A}}} \\ &\leq 2(2M_1)^\gamma k_{n+1}^{-\gamma/q_{\mathcal{A}}} \sum_{l \geq n+1} \left( \frac{k_l}{k_{n+1}} \right)^{-\gamma/q_{\mathcal{A}}}. \end{aligned}$$

Following the assumption  $k_{l+1} \geq a \cdot k_l$  for all  $l \geq 1$ , we get

$$\sum_{l \geq n} |X_{g_{l+1}(U)} - X_{g_l(U)}| + \sum_{l \geq n} |X_{g_{l+1}(V)} - X_{g_l(V)}| \leq 2(2M_1)^\gamma k_{n+1}^{-\gamma/q_{\mathcal{A}}} \sum_{l \geq 0} a^{-l \cdot \gamma/q_{\mathcal{A}}}. \quad (2.14)$$

Inequalities (2.12), (2.13) and (2.14) give

$$|X_U - X_V| \leq \left( 2M_1 k_{n+1}^{-1/q_{\mathcal{A}}} + d_{\mathcal{A}}(U, V) \right)^\gamma + 2(2M_1)^\gamma \frac{k_{n+1}^{-\gamma/q_{\mathcal{A}}}}{1 - a^{-\gamma/q_{\mathcal{A}}}}.$$

Choosing  $n \geq n^*$  such that  $M_1 k_{n+1}^{-1/q_{\mathcal{A}}} \leq d_{\mathcal{A}}(U, V) < M_1 k_n^{-1/q_{\mathcal{A}}}$ , we get

$$|X_U - X_V| \leq \left( 3^\gamma + \frac{2^{\gamma+1}}{1 - a^{-\gamma/q_{\mathcal{A}}}} \right) d_{\mathcal{A}}(U, V)^\gamma.$$

As a summary, we proved that there exist a constant  $L > 0$  and a random variable  $h^*$  such that

$$\forall U, V \in \mathcal{D}; \quad d_{\mathcal{A}}(U, V) < h^* \Rightarrow |X_U - X_V| \leq L d_{\mathcal{A}}(U, V)^\gamma \quad \text{a.s.} \quad (2.15)$$

In the last part of the proof, we need to extend (2.15) to the whole class  $\mathcal{A}$ . From Lemma 2.6, we can claim:

On  $\Omega^*$ , for all  $\epsilon \in (0, h^*)$ , for all  $U$  and  $V$  in  $\mathcal{A}$  with  $d_{\mathcal{A}}(U, V) < h^* - \epsilon$ , there exists  $n_0 > n^*$  such that  $d_{\mathcal{A}}(g_n(U), g_m(V)) < h^*$  for all  $n \geq n_0$  and  $m \geq n_0$ . Thus by (2.15),

$$\forall n > n_0, \forall m > n_0; \quad |X_{g_n(U)} - X_{g_m(V)}| \leq L d_{\mathcal{A}}(g_n(U), g_m(V))^\gamma. \quad (2.16)$$

We define the process  $\tilde{X}$  by

- $\forall \omega \notin \Omega^*, \forall U \in \mathcal{A}, \tilde{X}_U(\omega) = 0,$
- $\forall \omega \in \Omega^*,$ 
  - $\forall U \in \mathcal{D}, \tilde{X}_U(\omega) = X_U(\omega)$
  - $\forall U \in \mathcal{A} \setminus \mathcal{D}, \tilde{X}_U(\omega) = \lim_{n \rightarrow \infty} X_{g_n(U)}(\omega).$

Applying (2.16) with  $V = U$ , we can see that  $(X_{g_n(U)}(\omega))_{n \in \mathbf{N}}$  is a Cauchy sequence and then converges in  $\mathbf{R}$ .

The process  $\tilde{X}$  satisfies almost surely

$$\forall U, V \in \mathcal{A}; \quad d_{\mathcal{A}}(U, V) < h^* \Rightarrow |\tilde{X}_U - \tilde{X}_V| \leq L d_{\mathcal{A}}(U, V)^\gamma.$$

Moreover,

- $\forall U \in \mathcal{D}, \tilde{X}_U = X_U$  almost surely.

- $\forall U \in \mathcal{A} \setminus \mathcal{D}$ , by construction,  $X_{g_n(U)} \xrightarrow{\text{a.s.}} \tilde{X}_U$  as  $n \rightarrow \infty$ .

Since  $\mathbf{E}[|X_{g_n(U)} - X_U|^\alpha]$  converges to 0 when  $n \rightarrow \infty$ , the sequence  $(X_{g_n(U)})_{n \in \mathbf{N}}$  converges in probability to  $X_U$ . Then, there exists a subsequence converging almost surely.

From these two facts, we get  $\tilde{X}_U = X_U$  a.s.

□

As in the multiparameter's case, a simpler statement holds for Gaussian processes (see [28] for a detailed study of the Kolmogorov criterion in the multiparameter frame).

**Corollary 2.10.** *Let  $d_{\mathcal{A}}$  be a (pseudo-)distance on the indexing collection  $\mathcal{A}$ , whose subclasses  $\underline{\mathcal{A}} = (\mathcal{A}_n)_{n \in \mathbf{N}}$  satisfy Assumption 1. Let  $X = \{X_U; U \in \mathcal{A}\}$  be a centered Gaussian set-indexed process such that*

$$\forall U, V \in \mathcal{A}; \quad \mathbf{E}[|X_U - X_V|^2] \leq K d_{\mathcal{A}}(U, V)^{2\beta}$$

where  $K > 0$  and  $\beta > 0$ .

Then, the sample paths of  $X$  are almost surely locally  $\gamma$ -Hölder continuous for all  $\gamma \in (0, \beta)$ .

*Proof.* For any  $p \in \mathbf{N}^*$ , there exists a constant  $\lambda_p > 0$  such that for all centered Gaussian random variable  $Y$ , we have  $\mathbf{E}[Y^{2p}] = \lambda_p (\mathbf{E}[Y^2])^p$ . Then,

$$\forall U, V \in \mathcal{A}; \quad \mathbf{E}[|X_U - X_V|^{2p}] \leq K \lambda_p d_{\mathcal{A}}(U, V)^{2p\beta}.$$

For all  $\gamma \in (0, \beta)$ , there exists  $p \in \mathbf{N}^*$  such that  $2p\beta > q_{\underline{\mathcal{A}}}$ , where  $q_{\underline{\mathcal{A}}}$  is the discretization exponent of  $(\mathcal{A}_n)_{n \in \mathbf{N}}$ . By Theorem 2.9 the result follows. □

**Remark 2.11.** *The proof of Theorem 2.9 shows that when Condition (H2) is removed from Assumption 1, the conclusion remains true when the hypothesis (2.10) is strengthened in*

$$\forall U, V \in \mathcal{A}; \quad \mathbf{E}[|X_U - X_V|^\alpha] \leq K d_{\mathcal{A}}(U, V)^{2q_{\underline{\mathcal{A}}} + \beta}.$$

*In that case, the validity of Corollary 2.10 persists, since the integer  $p$  can be chosen such that  $2p\beta > 2q_{\underline{\mathcal{A}}} + \beta$  (instead of  $2p\beta > q_{\underline{\mathcal{A}}}$ ).*

As previously mentioned, the Brownian motion indexed by the lower layers of  $[0, 1]^2$  is discontinuous with probability one (e.g. see Theorem 1.4.5 in [5] or [2, 27]). Our Theorem 2.9 and Corollary 2.10 do not contradict this fact, since the collection of lower layers of  $[0, 1]^2$  do not satisfy Assumption 1 according to Lemma 2.4 in the specific case of the separating subclasses  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  mentioned there. This latter result is improved by the following corollary of Theorem 2.9.

**Corollary 2.12.** *Any subclasses  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  satisfying Condition (4) of Definition 2.1 for the indexing collection of lower layers of  $[0, 1]^2$  does not satisfy Assumption 1.*

Following the early work of Dudley, the restriction of the set-indexed Brownian motion to an indexing collection satisfying certain conditions can admit a continuous modification. We refer to [5] for a modern survey of these results. In particular, the set-indexed Brownian motion is continuous over any Vapnik-Červonenkis class of sets (see Corollary 1.4.10 in [5]), as the collection of rectangles of  $\mathbf{R}^N$  is an example.

**Remark 2.13.** According to Dudley's Theorem (see Theorem 2.7.1 of Chapter 5 in [28] and also Theorems 1.3.5 and 1.5.4 in [5]), the existence of a continuous modification of a centered Gaussian  $\mathcal{A}$ -indexed process can be proved if  $(\mathcal{A}, d_{\mathcal{A}})$  is totally bounded and  $\int_0^1 \sqrt{\log N(\mathcal{A}, \epsilon)} d\epsilon < +\infty$ , where  $N(\mathcal{A}, \epsilon)$  denotes the entropy function (relative to the distance  $d_{\mathcal{A}}$ ).

Following the continuity on SI processes indexed by Vapnik-Červonenkis classes and the role of Assumption 1 in Theorem 2.9, we emphasize the fact that upper bounds for the entropy function can be obtained in the two cases. Let us define

$$\forall n \in \mathbf{N}, \quad \phi(n) = M_1 k_{n+1}^{-1/q_{\mathcal{A}}}.$$

Let us also define, for  $\epsilon \in (0, \frac{1}{2}]$ ,  $n(\epsilon) = \inf\{k : \phi(k) < \epsilon\}$ . From Condition (H1) of Assumption 1, for all  $U \in \mathcal{A}$ ,

$$d_{\mathcal{A}}(U, g_{n(\epsilon)}(U)) \leq \phi(n(\epsilon)) \leq \epsilon,$$

which implies  $N(\mathcal{A}, \epsilon) \leq k_{n(\epsilon)}$ .

We can see easily that

$$0 < \epsilon \leq M_1 k_{n(\epsilon)}^{-1/q_{\mathcal{A}}},$$

which allows to get a bound for the entropy function (relative to the distance  $d_{\mathcal{A}}$ ),

$$N(\mathcal{A}, \epsilon) \leq k_{n(\epsilon)} \leq M_1^{q_{\mathcal{A}}} \epsilon^{-q_{\mathcal{A}}}. \quad (2.17)$$

In the case of a Vapnik-Červonenkis class  $\mathcal{D}$  of sets in a measure space  $(E, \mathcal{E}, \nu)$ , the entropy function (relative to the distance  $\nu(\bullet \Delta \bullet)$ ) is bounded as:

$$\forall 0 < \varepsilon \leq 1/2, \quad N(\mathcal{D}, \varepsilon) \leq K \varepsilon^{-2v} |\ln \varepsilon|^v, \quad (2.18)$$

where  $K$  and  $v$  are positive constants (e.g. see [5], Theorem 1.4.9).

So far we only considered simple increments of the process  $\{X_U; U \in \mathcal{A}\}$  of the form  $|X_U - X_V|$  for  $U, V \in \mathcal{A}$  not necessarily ordered. However these quantities do not constitute the natural extension of the one-parameter  $X_t - X_s$  ( $s, t \in \mathbf{R}_+$ ) to multiparameter (e.g. [28, 4, 20]) and set-indexed (e.g. [27, 24]) settings, particularly when increment stationarity property is concerned. The remaining of this section is devoted to usual increments of set-indexed processes. Let us define, for any given indexing collection  $\mathcal{A}$ , the collection  $\mathcal{C}$  of subsets of  $\mathcal{T}$ , defined as

$$\mathcal{C} = \{U_0 \setminus \cup_{i=1}^k U_i; U_0, U_1, \dots, U_k \in \mathcal{A}, k \in \mathbf{N}\}.$$

This collection is used to index the process  $\Delta X$ , defined by  $\Delta X_C = X_{U_0} - \Delta X_{U_0 \cap \cup_{i \geq 1} U_i}$  for  $C = U_0 \setminus \cup_{i=1}^k U_i$ , where  $\Delta X_{U_0 \cap \cup_{i \geq 1} U_i}$  is given by the inclusion-exclusion formula

$$\Delta X_{U_0 \cap \cup_{i \geq 1} U_i} = \sum_{i=1}^k \sum_{j_1 < \dots < j_i} (-1)^{i-1} X_{U_0 \cap U_{j_1} \cap \dots \cap U_{j_i}}. \quad (2.19)$$

The existence of the increment process  $\Delta X$  indexed by  $\mathcal{C}$  assume that for any  $C \in \mathcal{C}$ , the value  $\Delta X_C$  does not depend on the representation of  $C$ .

**Corollary 2.14.** *Under the hypotheses of Theorem 2.9 and if the distance  $d_{\mathcal{A}}$  on the class  $\mathcal{A}$  is assumed to be contracting, for each fixed integer  $l \geq 1$ , for all  $\gamma \in (0, \beta/\alpha)$ , there exist a random variable  $h^{**}$  and a constant  $L > 0$  such that, with probability one,*

$$\forall C = U \setminus \bigcup_{i \leq l} V_i \text{ with } U, V_1, \dots, V_l \in \mathcal{A},$$

$$\max_{i \leq l} \{m(U \setminus V_i)\} < h^{**} \Rightarrow |\Delta X_C| \leq L m(C)^\gamma. \quad (2.20)$$

In order to prove Corollary 2.14, we need the following lemma:

**Lemma 2.15.** *If the distance  $d_{\mathcal{A}}$  on the class  $\mathcal{A}$  is contracting, then for  $U, V_1, V_2 \in \mathcal{A}$ ,*

$$d_{\mathcal{A}}(U, V_1) \vee d_{\mathcal{A}}(U, V_2) \leq \rho \Rightarrow d_{\mathcal{A}}(U, V_1 \cap V_2) \leq 3\rho.$$

Moreover, for any integer  $l \geq 1$  and for all  $U, V_1, \dots, V_l \in \mathcal{A}$ ,

$$\max_{i \leq l} \{d_{\mathcal{A}}(U, V_i)\} \leq \rho \Rightarrow d_{\mathcal{A}}(U, V_1 \cap \dots \cap V_l) \leq K(l) \rho,$$

for some constant  $K(l) > 0$  which only depends on  $l$ .

*Proof of Lemma 2.15.* The proof relies on the triangular inequality and the contracting property of  $d_{\mathcal{A}}$ .  $\square$

*Proof of Corollary 2.14.* Assuming that  $g_n$  can be extended to  $\mathcal{A}(u)$  in the following way

$$\forall V_1, \dots, V_p \in \mathcal{A}, \quad g_n\left(\bigcup_{i=1}^p V_i\right) = \bigcup_{i=1}^p g_n(V_i)$$

and since for all  $V_1, \dots, V_p \in \mathcal{A}$ ,

$$\Delta X_{\cup V_i} = \sum_{i=1}^p X_{V_i} + \dots + (-1)^{k-1} \sum_{i_1 < \dots < i_k} X_{\cap_{i=1}^k V_i} + \dots + (-1)^{p-1} X_{V_1 \cap \dots \cap V_p},$$

one can express

$$\begin{aligned} |\Delta X_{g_{n+1}(\cup V_i)} - \Delta X_{g_n(\cup V_i)}| &\leq \sum_{i=1}^p |X_{g_{n+1}(V_i)} - X_{g_n(V_i)}| + \dots \\ &\quad + \sum_{i_1 < \dots < i_k} |X_{g_{n+1}(\cap_{i=1}^k V_i)} - X_{g_n(\cap_{i=1}^k V_i)}| + \dots \\ &\quad + |X_{g_{n+1}(\cap_{i=1}^p V_i)} - X_{g_n(\cap_{i=1}^p V_i)}|. \end{aligned} \quad (2.21)$$

From Theorem 2.9, there exist a random variable  $h^*$  and a constant  $L > 0$  such that almost surely, for all  $n \in \mathbb{N}$  satisfying  $2M_1 k_{n+1}^{-1/q_{\mathcal{A}}} < h^*$ ,

$$\forall U \in \mathcal{A}, \quad |X_{g_n(U)} - X_{g_{n+1}(U)}| < L d_m(g_n(U), g_{n+1}(U))^\gamma \leq L (2M_1)^\gamma k_{n+1}^{-\gamma/q_{\mathcal{A}}}.$$

When  $p \leq l$ , the number of terms in the right side of inequality (2.21) is bounded by a constant, independent of the set  $V_1, \dots, V_p \in \mathcal{A}$ . Thus, there exists a positive constant  $K_2(l)$  such that

$$|\Delta X_{g_{n+1}(\cup V_i)} - \Delta X_{g_n(\cup V_i)}| \leq K_2(l) k_{n+1}^{-\gamma/q_{\mathcal{A}}}. \quad (2.22)$$

Then, assuming  $M_1 k_{n_0+1}^{-1/q_A} \leq \max_{i \leq l} \{m(U \setminus V_i)\} < M_1 k_{n_0}^{-1/q_A}$  for  $U, V_1, \dots, V_l \in \mathcal{A}$  and  $2M_1 k_{n_0}^{-1/q_A} \leq h^*$ , the following inequality holds:

$$\begin{aligned} |X_U - \Delta X_{\cup V_i}| &\leq |X_{g_{n_0}(U)} - \Delta X_{g_{n_0}(\cup V_i)}| + \sum_{j \geq n_0} |X_{g_{j+1}(U)} - X_{g_j(U)}| \\ &\quad + \sum_{j \geq n_0} |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}|. \end{aligned} \quad (2.23)$$

According to Assumption 1, there exists a real  $a > 1$  such that  $\forall n, k_{n+1} \geq a.k_n$ . Then the last two terms are bounded from above using (2.22):

$$\begin{aligned} \sum_{j \geq n_0} |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}| &\leq \sum_{j \geq n_0+1} K_2(l) k_j^{-\gamma/q_A} \leq K_2(l) k_{n_0+1}^{-\gamma/q_A} \sum_{j \geq 0} a^{j\gamma/q_A} \\ &\leq K_3(l, \gamma, q_A) m(C)^\gamma, \end{aligned}$$

using the fact that  $m(C) \geq \max_{i \leq l} m(U \setminus V_i)$ .

In the same way, the second term of the upper bound (2.23) is proved to be bounded by  $K_4(\gamma, q_A) m(C)^\gamma$ , where  $K_4(\gamma, q_A) > 0$  only depends on  $\gamma$  and  $q_A$ .

The first term of (2.23) can be bounded by a finite sum (whose number of terms only depends on  $l$ ) of the form  $|X_{g_{n_0}(U)} - X_{g_{n_0}(V_{i_1, \dots, i_k})}|$ , where  $V_{i_1, \dots, i_k} = V_{i_1} \cap \dots \cap V_{i_k}$  for  $i_1 < \dots < i_k \leq l$ :

$$|X_{g_{n_0}(U)} - \Delta X_{g_{n_0}(\cup V_i)}| \leq \sum_{j=1}^l \sum_{i_1 < \dots < i_j} |X_{g_{n_0}(U)} - X_{g_{n_0}(V_{i_1, \dots, i_j})}|. \quad (2.24)$$

Condition (H1) of Assumption 1 and Lemma 2.15 imply

$$\begin{aligned} d_m(g_{n_0}(U), g_{n_0}(V_{i_1, \dots, i_j})) &\leq d_m(g_{n_0}(U), U) + d_m(U, V_{i_1, \dots, i_j}) + d_m(V_{i_1, \dots, i_j}, g_{n_0}(V_{i_1, \dots, i_j})) \\ &\leq K(l) \max_{i \leq l} \{m(U \setminus V_i)\} + 2M_1 k_{n_0+1}^{-1/q_A} \\ &\leq (K(l) + 2) \max_{i \leq l} \{m(U \setminus V_i)\}. \end{aligned}$$

Hence, Theorem 2.9 implies that when  $\max_{i \leq l} \{m(U \setminus V_i)\} < (K(l) + 2)^{-1} h^*$ , each term of equation (2.24) is bounded by a quantity proportional to  $m(C)^\gamma$ . Then, the random variable  $h^{**}$  of the statement can be chosen to be  $(K(l) + 2)^{-1} h^*$  and the result follows.  $\square$

Corollary 2.14, as a result on the class  $\mathcal{C}^l = \{U \setminus V; U \in \mathcal{A}, V \in \mathcal{B}^l\}$  where  $\mathcal{B}^l = \{\bigcup_{i=1}^l V_i; V_1, \dots, V_l \in \mathcal{A}\}$ , does *not* extend to the whole  $\mathcal{C} = \bigcup_{l \geq 1} \mathcal{C}^l$ , as the following example shows. The next result is an adaptation of an example in [5, 27] to the set-indexed setting. It states that the Brownian motion can be unbounded on  $\mathcal{C}$  when  $\mathcal{A}$  is the collection of rectangles of  $[0, 1]^2$ .

**Proposition 2.16.** *Let  $W$  be a Brownian motion indexed by the Borelian sets of  $[0, 1]^2$ , i.e. a centered Gaussian process with covariance structure*

$$\mathbf{E}[W_C W_{C'}] = \lambda(C \cap C'), \quad \forall C, C' \in \mathcal{B}([0, 1]^2)$$

where  $\lambda$  denotes the Lebesgue measure.

Let  $\mathcal{A}$  be the collection of rectangles of  $[0, 1]^2$ . In the sequel, we consider the restriction



on the class  $\mathcal{C}$ , related to  $\mathcal{A}$ , of the Brownian motion defined above.

Then for all  $h > 0$ , all  $M > 0$ , and for almost all  $\omega \in \Omega$ , there exist sequences of sets  $(C_n(\omega))_{n \in \mathbf{N}}$ ,  $(C'_n(\omega))_{n \in \mathbf{N}}$  in  $\mathcal{C}$  such that  $\lambda(C_n(\omega)) \vee \lambda(C'_n(\omega)) < h$  and for  $n$  big enough,

$$\max\{|W_{C_n(\omega)}(\omega)|, |W_{C'_n(\omega)}(\omega)|\} > \frac{M}{8}.$$

Without any stronger condition than Assumption 1 on the sub-semilattices  $(\mathcal{A}_n)_{n \in \mathbf{N}}$ , the previous example of set-indexed Brownian motion dismisses a possible definition of the Hölder continuity for stochastic processes of the form:

$$\exists M > 0, \exists \delta_0 > 0 : \forall C \in \mathcal{C} \text{ with } m(C) < \delta_0, |\Delta X_C| \leq M.m(C)^\alpha.$$

### 3. HÖLDER EXPONENTS FOR SET-INDEXED PROCESSES

For a function  $f : \mathbf{R}_+^N \mapsto \mathbf{R}$ , which is continuous but not differentiable, we usually consider two kinds of Hölder exponent at  $t_0 \in \mathbf{R}_+^N$ :

- the pointwise Hölder exponent

$$\alpha_f(t_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{s, t \in B(t_0, \rho)} \frac{|f(t) - f(s)|}{\rho^\alpha} < \infty \right\},$$

- and the local Hölder exponent

$$\tilde{\alpha}_f(t_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{s, t \in B(t_0, \rho)} \frac{|f(t) - f(s)|}{\|t - s\|^\alpha} < \infty \right\}.$$

Each one allows to measure the regularity of the function  $f$ . In general, we have  $\tilde{\alpha}_f \leq \alpha_f$  but the inequality can be strict.

**Example 3.1.** Let  $\gamma > 0, \delta > 0$ . Let  $f$  be a chirp function, i.e.  $t \mapsto |t|^\gamma \sin \frac{1}{|t|^\delta}$ . The two Hölder exponents at 0 can be computed and  $\tilde{\alpha}(0) = \frac{\gamma}{1+\delta} < \alpha(0) = \gamma$ .

This example shows that the sole pointwise exponent is not sufficient to describe the irregularity of the function. The local exponent *can see* the oscillation around 0, while the pointwise exponent cannot. These two notions can be applied to study regularity of paths of a stochastic process.

In the case of Gaussian processes (see [22]), we define the *deterministic pointwise Hölder exponent*

$$\omega_X(t_0) = \sup \left\{ \sigma; \limsup_{\rho \rightarrow 0} \sup_{s, t \in B(t_0, \rho)} \frac{\mathbf{E}[X_t - X_s]^2}{\rho^{2\sigma}} < \infty \right\} \quad (3.1)$$

and the *deterministic local Hölder exponent*

$$\tilde{\omega}_X(t_0) = \sup \left\{ \sigma; \limsup_{\rho \rightarrow 0} \sup_{s, t \in B(t_0, \rho)} \frac{\mathbf{E}[X_t - X_s]^2}{\|t - s\|^{2\sigma}} < \infty \right\}. \quad (3.2)$$

In [22], it is shown that for all  $t_0 \in \mathbf{R}_+^N$ , the pointwise and local Hölder exponents of  $X$  at  $t_0$  satisfy almost surely

$$\alpha_X(t_0) = \omega_X(t_0) \quad \text{and} \quad \tilde{\alpha}_X(t_0) = \tilde{\omega}_X(t_0).$$

In the three following sections, several definitions are studied for Hölder regularity of set-indexed processes. They are connected to the various ways to study the local behaviour of the sample paths of  $X$  around a given set  $U_0 \in \mathcal{A}$ .

**3.1. Definition of Hölder exponents for set-indexed processes.** Back to the beginning of Section 2.2, localizing the two expressions (i.) and (ii.) for Hölder-continuity leads to two different notions. Indeed, for the distance  $d_{\mathcal{A}}$  on  $\mathcal{A}$ , if  $B_{d_{\mathcal{A}}}(U_0, \rho)$  denotes the open ball centered in  $U_0 \in \mathcal{A}$  and whose radius is  $\rho > 0$ , we get

(i.)<sub>loc</sub>

$$\limsup_{\delta \rightarrow 0+} \delta^{-q} \sup_{U, V \in B_{d_{\mathcal{A}}}(U_0, \delta)} |f(U) - f(V)| < \infty.$$

(ii.)<sub>loc</sub> There exist  $M > 0$  and  $\delta_0 > 0$  such that

$$\forall U, V \in B_{d_{\mathcal{A}}}(U_0, \delta_0); \quad |f(U) - f(V)| \leq M d_{\mathcal{A}}(U, V)^q.$$

Although the conditions (i.) and (ii.) are equivalent, localizing around  $U_0 \in \mathcal{A}$  only gives (ii.)<sub>loc</sub>  $\Rightarrow$  (i.)<sub>loc</sub>.

In the same way as in the multiparameter case, it is natural to define the *pointwise Hölder exponent* of a set-indexed process  $X$  at  $U_0 \in \mathcal{A}$ ,

$$\alpha_X(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{U, V \in B_{d_{\mathcal{A}}}(U_0, \rho)} \frac{|X_U - X_V|}{\rho^\alpha} < \infty \right\}, \quad (3.3)$$

and the *local Hölder exponent* at  $U_0 \in \mathcal{A}$ ,

$$\tilde{\alpha}_X(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{U, V \in B_{d_{\mathcal{A}}}(U_0, \rho)} \frac{|X_U - X_V|}{d_{\mathcal{A}}(U, V)^\alpha} < \infty \right\}. \quad (3.4)$$

As in the deterministic case, we have in general

$$\forall \omega \in \Omega, \forall U_0 \in \mathcal{A}; \quad \tilde{\alpha}_X(U_0)(\omega) \leq \alpha_X(U_0)(\omega). \quad (3.5)$$

**Remark 3.2.** We can see that condition (i.)<sub>loc</sub> is equivalent to  $q < \alpha(U_0)$ , and condition (ii.)<sub>loc</sub> is equivalent to  $q < \tilde{\alpha}(U_0)$ . Then (3.5) is another statement for (ii.)<sub>loc</sub>  $\Rightarrow$  (i.)<sub>loc</sub>.

**3.2. Definition of Hölder exponents on  $\mathcal{C}^l$ .** Following expression (2.19) for the definition of the increments of a set-indexed process, we consider alternative definitions for Hölder exponents, where the quantities  $X_U - X_V$  are substituted with  $\Delta X_{U \setminus V}$ .

As stated at the end of Section 2.2, it is not wise to consider  $\Delta X_{U \setminus V}$  when  $U \in \mathcal{A}$  and  $V \in \mathcal{A}(u)$  are close to a given  $U_0 \in \mathcal{A}$ . Indeed, Proposition 2.16 shows that the quantity  $|\Delta X_{U \setminus V}|$  can stay far away from 0 when  $m(U \setminus V)$  is small, even in the simple case of a Brownian motion. However, when  $U \in \mathcal{A}$  and  $V$  is restricted to sets of the form  $V = \bigcup_{1 \leq i \leq l} V_i$  where  $l$  is fixed and  $V_1, \dots, V_l \in \mathcal{A}$ , the Hölder regularity can be defined from the study of  $\Delta X_{U \setminus V}$ .

Fix any integer  $l \geq 1$  and set for all  $U \in \mathcal{A}$  and  $\rho > 0$ ,

$$\mathcal{B}^l(U, \rho) = \left\{ \bigcup_{1 \leq i \leq l} V_i; \ V_1, \dots, V_l \in \mathcal{A}, \ \max_{1 \leq i \leq l} d_{\mathcal{A}}(U, V_i) < \rho \right\}.$$

The pointwise and local Hölder  $\mathcal{C}^l$ -exponents at  $U_0 \in \mathcal{A}$  are respectively defined as

$$\alpha_{X,\mathcal{C}^l}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ V \in \mathcal{B}^l(U_0, \rho)}} \frac{|\Delta X_{U \setminus V}|}{\rho^\alpha} < \infty \right\},$$

and

$$\tilde{\alpha}_{X,\mathcal{C}^l}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ V \in \mathcal{B}^l(U_0, \rho)}} \frac{|\Delta X_{U \setminus V}|}{d_{\mathcal{A}}(U, V)^\alpha} < \infty \right\}.$$

The following result shows that the  $\mathcal{C}^l$ -exponents do not depend on  $l$  and, consequently, they provide a definition of Hölder exponents on the class  $\mathcal{C}$ . Moreover, these exponents can be compared to the exponents defined by (3.3) and (3.4).

**Proposition 3.3.** *If  $d_{\mathcal{A}}$  is a contracting distance, for any  $U_0 \in \mathcal{A}$ , the exponents  $\alpha_{X,\mathcal{C}^l}(U_0)$  and  $\tilde{\alpha}_{X,\mathcal{C}^l}(U_0)$  do not depend on the integer  $l \geq 1$ . They are denoted by  $\alpha_{X,\mathcal{C}}(U_0)$  and  $\tilde{\alpha}_{X,\mathcal{C}}(U_0)$  respectively.*

Moreover, for all  $U_0 \in \mathcal{A}$  and all  $\omega \in \Omega$ ,

$$\alpha_{X,\mathcal{C}}(U_0)(\omega) \geq \alpha_X(U_0)(\omega) \quad \text{and} \quad \tilde{\alpha}_{X,\mathcal{C}}(U_0)(\omega) \geq \tilde{\alpha}_X(U_0)(\omega).$$

*Proof.* We only detail the case of the pointwise exponent. The proof for the local exponent is totally similar.

From the definition of the  $\mathcal{C}^l$ -exponents, since  $l \geq l'$  implies  $\mathcal{B}^{l'}(U_0, \rho) \subseteq \mathcal{B}^l(U_0, \rho)$ , it is clear that

$$\forall \omega \in \Omega, \forall l \geq l', \quad \alpha_{X,\mathcal{C}^l}(U_0)(\omega) \leq \alpha_{X,\mathcal{C}^{l'}}(U_0)(\omega).$$

For sake of readability, we prove the converse inequality for  $l = 2, l' = 1$  (the other cases are very similar). For any  $\rho > 0$ , let  $U \in B_{d_{\mathcal{A}}}(U_0, \rho)$ , and  $V = V_1 \cup V_2 \in \mathcal{B}^1(U_0, \rho)$  with  $V_1, V_2 \in \mathcal{A}$ . From the inclusion-exclusion formula,

$$\begin{aligned} |\Delta X_{U \setminus V}| &= |X_U - X_{U \cap V_1} - X_{U \cap V_2} + X_{U \cap V_1 \cap V_2}| \\ &= |\Delta X_{U \setminus V_1} + \Delta X_{U \setminus V_2} - \Delta X_{U \setminus (V_1 \cap V_2)}| \\ &\leq |\Delta X_{U \setminus V_1}| + |\Delta X_{U \setminus V_2}| + |\Delta X_{U \setminus (V_1 \cap V_2)}|. \end{aligned}$$

We have  $d_{\mathcal{A}}(U_0, V_1) \leq \rho$ ,  $d_{\mathcal{A}}(U_0, V_2) \leq \rho$  and

$$\begin{aligned} d_{\mathcal{A}}(U_0, V_1 \cap V_2) &\leq d_{\mathcal{A}}(U_0, V_1) + d_{\mathcal{A}}(V_1, V_1 \cap V_2) \\ &\leq d_{\mathcal{A}}(U_0, V_1) + d_{\mathcal{A}}(V_1, V_2) \leq 2d_{\mathcal{A}}(U_0, V_1) + d_{\mathcal{A}}(U_0, V_2) \leq 3\rho, \end{aligned}$$

using  $d_{\mathcal{A}}(V_1, V_1 \cap V_2) \leq d_{\mathcal{A}}(V_1, V_2)$  from the contracting property of  $d_{\mathcal{A}}$ .

Then, for all  $\alpha < \alpha_{X,\mathcal{C}^{l'}}(U_0)(\omega)$ ,

$$\limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ V \in \mathcal{B}^l(U_0, \rho)}} \frac{|\Delta X_{U \setminus V}|}{\rho^\alpha} < \infty,$$

which says that  $\alpha < \alpha_{X,\mathcal{C}^l}(U_0)(\omega)$ . Thus,  $\alpha_{X,\mathcal{C}^{l'}}(U_0)(\omega) \leq \alpha_{X,\mathcal{C}^l}(U_0)(\omega)$ .

This inequality achieves to prove that  $\alpha_{X,\mathcal{C}^l}(U_0)(\omega)$  does not depend on the integer  $l \geq 1$ .

To prove the second part of the Proposition, it suffices then to prove the inequality for  $l = 1$ . This is straightforward, since for a fixed  $U \in B_{d_A}(U_0, \rho)$ ,

$$\sup_{V \in \mathcal{B}^1(U_0, \rho)} |\Delta X_{U \setminus V}| \leq \sup_{W \in B_{d_A}(U_0, \rho)} |X_U - X_W|.$$

Hence  $\alpha_X(U_0) \leq \alpha_{X, \mathcal{C}^1}(U_0)$ . The inequality for the local exponent can be obtained identically, or one can notice that it is a direct consequence of Corollary 2.14.

The converse inequality does not hold in general since quantities  $|X_U - X_V|$  cannot be obtained from the increment process  $\Delta X$  when  $U, V$  are not ordered.  $\square$

**Remark 3.4.** *The previous definition of the pointwise Hölder exponent on  $\mathcal{C}^l$  is not equivalent to the quantity*

$$\sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_A}(U_0, \rho) \\ V \in \mathcal{B}^l: d_A(U_0, V) < \rho}} \frac{|\Delta X_{U \setminus V}|}{\rho^\alpha} < \infty \right\},$$

as the following example shows.

In the particular case of the indexing collection  $\mathcal{A}$  equal to the rectangles of  $\mathbf{R}_+^2$  and the distance  $d_\lambda = \lambda(\bullet \triangle \bullet)$  induced by the Lebesgue measure  $\lambda$  of  $\mathbf{R}^2$ , we show that the assertion  $(V \in \mathcal{B}^1 : d_\lambda(U_0, V) < \rho)$  is not equivalent to  $(V \in \mathcal{B}^1(U_0, \rho))$ .

Consider  $V_1 = [0; (n^2, n^2 + \frac{1}{n})]$ ,  $V_2 = [0; (n^2 + \frac{1}{n}, n^2)]$  and  $U = [0; (n^2 + \frac{1}{n}, n^2 + \frac{1}{n})]$ . We have

$$d_\lambda(U, V_1 \cup V_2) = \frac{1}{n^2} \quad \text{while} \quad d_\lambda(U, V_1) = d_\lambda(U, V_2) \approx n.$$

Then,  $V_1 \cup V_2 \notin \mathcal{B}^2(U, \rho)$  for small  $\rho$  and it is not possible to control the quantity  $|X_U - \Delta X_{V_1 \cup V_2}|$  using  $|X_U - X_{V_1}|$ ,  $|X_U - X_{V_2}|$  and  $|X_U - X_{V_1 \cap V_2}|$  as was done in the previous proofs.

The notation  $\alpha_{X, \mathcal{C}}$  must be considered with precaution: Proposition 2.16 shows that the Hölder exponents cannot be defined directly by taking the supremum on  $U \in \mathcal{A}$  and  $V \in \mathcal{A}(u)$  with  $d_A(U_0, U) < \rho$  and  $d_A(U_0, V) < \rho$  (and then, on the class  $\mathcal{C}$ ). This is the reason why the set  $V$  is restricted to be in  $\mathcal{B}^l(U, \rho)$ .

The arguments of the proof of Proposition 3.3 in the particular case of  $l = 1$  leads to: for all  $\omega$ ,

$$\alpha_{X, \mathcal{C}}(U_0)(\omega) \geq \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U, V \in B_{d_A}(U_0, \rho) \\ U \subset V}} \frac{|X_U(\omega) - X_V(\omega)|}{\rho^\alpha} < \infty \right\},$$

and

$$\tilde{\alpha}_{X, \mathcal{C}}(U_0)(\omega) \geq \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U, V \in B_{d_A}(U_0, \rho) \\ U \subset V}} \frac{|X_U(\omega) - X_V(\omega)|}{d_A(U, V)^\alpha} < \infty \right\}.$$

The converse inequalities follow from the fact that the set of  $U, V \in B_{d_A}(U_0, \rho)$  with  $U \subset V$  is included in the set of  $U \in B_{d_A}(U_0, \rho)$  and  $V \in \mathcal{B}^1(U_0, \rho)$ . Then, we can state:

**Corollary 3.5.** *If  $d_{\mathcal{A}}$  is a contracting distance, the pointwise and local Hölder  $\mathcal{C}$ -exponents are respectively given by*

$$\alpha_{X,\mathcal{C}}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U, V \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ U \subset V}} \frac{|X_U - X_V|}{\rho^\alpha} < \infty \right\},$$

and

$$\tilde{\alpha}_{X,\mathcal{C}}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U, V \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ U \subset V}} \frac{|X_U - X_V|}{d_{\mathcal{A}}(U, V)^\alpha} < \infty \right\}.$$

**3.3. Pointwise continuity.** We give here a weak form of continuity which is particularly relevant when dealing with set-indexed Poisson process, set-indexed Brownian motion and more generally set-indexed Lévy processes (see [25]). Such a definition does not even require Assumption 1 on  $\mathcal{A}$ . Following [25], we define:

**Definition 3.6.** *The point mass jump of a set-indexed function  $x : \mathcal{A} \rightarrow \mathbf{R}$  at  $t \in \mathcal{T}$  is defined by*

$$J_t(x) = \lim_{n \rightarrow \infty} \Delta x_{C_n(t)}, \quad \text{where } C_n(t) = \bigcap_{\substack{C \in \mathcal{C}_n \\ t \in C}} C \quad (3.6)$$

and for each  $n \geq 1$ ,  $\mathcal{C}_n$  denotes the collection of subsets  $U \setminus V$  with  $U \in \mathcal{A}_n$  and  $V \in \mathcal{A}_n(u)$ .

**Definition 3.7.** *A set-indexed function  $x : \mathcal{A} \rightarrow \mathbf{R}$  is said pointwise continuous at  $t \in \mathcal{T}$  if  $J_t(x) = 0$ .*

Let us recall that a subset  $\mathcal{A}'$  of  $\mathcal{A}$  which is closed under arbitrary intersections is called a *lower sub-semilattice* of  $\mathcal{A}$ . The ordering of a lower sub-semilattice  $\mathcal{A}' = \{A_1, A_2, \dots\}$  is said to be *consistent* if  $A_i \subset A_j \Rightarrow i \leq j$ . Proceeding inductively, we can show that any lower sub-semilattice admits a consistent ordering, which is not unique in general (see [27]).

If  $\{A_1, \dots, A_n\}$  is a consistent ordering of a finite lower sub-semilattice  $\mathcal{A}'$ , the set  $C_i = A_i \setminus \bigcup_{j \leq i-1} A_j$  is called *the left neighbourhood* of  $A_i$  in  $\mathcal{A}'$ . Since  $C_i = A_i \setminus \bigcup_{A \in \mathcal{A}', A \not\subset A_i} A$ , the definition of the left neighbourhood does not depend on the ordering.

As in the classical Kolmogorov criterion of continuity, the pointwise continuity of a set-indexed process  $X$  can be proved from the study of  $\mathbf{E}[|\Delta X_{C_n(t)}|^p]$  when  $n$  goes to infinity.

**Proposition 3.8.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be a set-indexed process such that, for some fixed  $t \in \mathcal{T}$ , there exists positive constants  $p, q$  and  $K$  such that for all  $n \in \mathbf{N}$ ,*

$$\mathbf{E} [|\Delta X_{C_n(t)}|^p] \leq K m(C_n(t))^q. \quad (3.7)$$

*Then, there exists an increasing function  $\varphi : \mathbf{N} \rightarrow \mathbf{N}$  such that, for all  $\gamma \in (0, \frac{q}{p})$ , there exists a random variable  $n^* > 0$  such that:*

$$\forall n \geq n^*, \quad |\Delta X_{C_{\varphi(n)}(t)}| \leq m(C_{\varphi(n)}(t))^\gamma \quad \text{a.s.}$$

*Proof.* Fix  $t \in \mathcal{T}$  and  $\gamma \in (0, \frac{q}{p})$ . For all  $n \in \mathbf{N}$ , inequality (3.7) implies

$$\begin{aligned} \mathbf{P}(|\Delta X_{C_n(t)}| > m(C_n(t))^\gamma) &\leq \frac{\mathbf{E}(|\Delta X_{C_n(t)}|^p)}{m(C_n(t))^{p\gamma}} \\ &\leq K m(C_n(t))^{q-p\gamma}. \end{aligned}$$

We consider an increasing function  $\varphi : \mathbf{N} \rightarrow \mathbf{N}$  such that  $m(C_{\varphi(n+1)}(t)) \geq 2 m(C_{\varphi(n)}(t))$  for all  $n \in \mathbf{N}$ . Hence, whenever  $\gamma < q/p$ , Borel-Cantelli Lemma implies the existence of an  $\mathbf{N}$ -valued random variable  $n^* \geq N$  such that, almost surely,

$$\forall n \geq n^*, \quad |\Delta X_{C_{\varphi(n)}(t)}| \leq m(C_{\varphi(n)}(t))^\gamma.$$

□

In the Gaussian case, the result of Proposition 3.8 can be improved: a uniform result can be obtained when the points  $t$  are restricted to be in a compact subset of  $\mathcal{T}$ .

**Proposition 3.9.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be a centered Gaussian set-indexed process and let  $U_{\max}$  be a subset in  $\mathcal{A}$  such that  $m(U_{\max}) < +\infty$  and, for all  $t \in U_{\max}$  and all  $n \geq N$ ,*

$$\mathbf{E}(|\Delta X_{C_n(t)}|^2) \leq K m(C_n(t))^{2q} \quad (3.8)$$

for constants  $q > 0$ ,  $K > 0$  and  $N \geq 1$ .

Then, there exists an increasing function  $\varphi : \mathbf{N} \rightarrow \mathbf{N}$  such that for any  $0 < \gamma < q$ , there exists a random variable  $n^* > 0$  satisfying, with probability one,

$$\forall t \in U_{\max}, \forall n \geq n^*, \quad |\Delta X_{C_{\varphi(n)}(t)}| \leq m(C_{\varphi(n)}(t))^\gamma.$$

*Proof.* Up to restricting the indexing collection to  $\{U \cap U_{\max}; U \in \mathcal{A}\}$ , we assume in this proof that the indexing collection  $\mathcal{A}$  is included in  $U_{\max}$ . For all  $0 < \gamma < q$ , we consider  $S_n = \sup \left\{ \frac{|\Delta X_{C_n(t)}|}{m(C_n(t))^\gamma}; t \in U_{\max} \right\}$ , where  $C_n(t)$  is defined in (3.6). When  $t$  ranges  $U_{\max}$ , the subset  $C_n(t)$  ranges  $\mathcal{C}^l(\mathcal{A}_n)$ , the collection of the disjoint left-neighbourhoods of  $\mathcal{A}_n$ . Consequently we can write  $S_n = \sup \left\{ \frac{|\Delta X_C|}{m(C)^\gamma}; C \in \mathcal{C}^l(\mathcal{A}_n) \right\}$ .

For any integer  $p \geq 1$ , we have

$$\begin{aligned} \mathbf{P}(S_n \geq 1) &\leq \sum_{C \in \mathcal{C}^l(\mathcal{A}_n)} \mathbf{P}(|\Delta X_C| \geq m(C)^\gamma) \\ &\leq \sum_{C \in \mathcal{C}^l(\mathcal{A}_n)} \frac{\mathbf{E}[|\Delta X_C|^{2p}]}{m(C)^{2\gamma p}} \leq K \lambda_p \sum_{C \in \mathcal{C}^l(\mathcal{A}_n)} m(C)^{2p(q-\gamma)}, \end{aligned}$$

where  $\lambda_p$  denotes the positive constant such that  $\mathbf{E}[Y^{2p}] = \lambda_p \mathbf{E}[Y^2]^p$  for all centered Gaussian random variable  $Y$ .

If  $p$  is chosen such that  $2p(q-\gamma) > 1$ , we have

$$\begin{aligned} \mathbf{P}(S_n \geq 1) &\leq K \lambda_p \left( \sum_{C \in \mathcal{C}^l(\mathcal{A}_n)} m(C) \right) \sup_{C \in \mathcal{C}^l(\mathcal{A}_n)} \{m(C)^{2p(q-\gamma)-1}\} \\ &\leq \lambda_p K m(U_{\max}) \sup_{C \in \mathcal{C}^l(\mathcal{A}_n)} \{m(C)^{2p(q-\gamma)-1}\} \end{aligned}$$

where the fact that the  $C \in \mathcal{C}^l(\mathcal{A}_n)$  are disjoint is used. For an appropriate  $\varphi$  (for instance the one chosen in the previous proof),  $\sup_{C \in \mathcal{C}^l(\mathcal{A}_{\varphi(n)})} m(C)$  is summable. Hence the Borel-Cantelli Lemma implies that for  $0 < \gamma < q$ ,  $\{S_{\varphi(n)} < 1\}$  happens infinitely often, which gives the result.  $\square$

**Remark 3.10.** Propositions 3.8 and 3.9 do not require Assumption 1 for the collection  $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$  and the distance  $d_m$ .

From these propositions, it is natural to define local Hölder regularity of a set-indexed process by a comparison of  $\Delta X_{C_n(t)}$  to quantities  $m(C_n(t))^\alpha$  with  $\alpha > 0$ , when  $n$  is big.

**Definition 3.11.** The pointwise continuity Hölder exponent at any  $t \in \mathcal{T}$  is defined by

$$\alpha_X^{pc}(t) = \sup \left\{ \alpha : \limsup_{n \rightarrow \infty} \frac{|\Delta X_{C_n(t)}|}{m(C_n(t))^\alpha} < \infty \right\}.$$

According to Proposition 3.8, if  $X$  is a  $\mathcal{A}$ -indexed process satisfying equation (3.7) for some  $t \in \mathcal{T}$ , then  $\alpha_X^{pc}(t) \geq \frac{q}{p}$  almost surely.

**Example 3.12.** As previously mentioned, the set-indexed Brownian motion can be not continuous, when the indexing collection is not a Vapnik-Červonenkis class (see [5, 27] for the detailed study). However, in [25], its sample paths are proved to be pointwise continuous as a set-indexed Lévy process with Gaussian increments.

#### 4. CONNECTION WITH HÖLDER EXPONENTS OF PROJECTIONS ON FLOWS

As the projection of a set-indexed process on any flow is a real-parameter process, its classical Hölder exponents can be considered and compared to the exponents of the set-indexed process.

In this section, we consider the concept of *flow*, which is a useful tool to reduce characterization or convergence problems to a one-dimensional issue. *Flows* have been used to characterize: strong martingales ([27]), set-Markov processes ([9]), set-indexed fractional Brownian motion ([24]) and set-indexed Lévy processes ([25]).

**Definition 4.1.** An elementary flow is defined to be a continuous increasing function  $f : [a, b] \subset \mathbf{R}_+ \rightarrow \mathcal{A}$ , i.e. such that

$$\begin{aligned} \forall s, t \in [a, b]; \quad s < t &\Rightarrow f(s) \subseteq f(t) \\ \forall s \in [a, b); \quad f(s) &= \bigcap_{v > s} f(v) \\ \forall s \in (a, b); \quad f(s) &= \overline{\bigcup_{u < s} f(u)}. \end{aligned}$$

A simple flow is a continuous function  $f : [a, b] \rightarrow \mathcal{A}(u)$  such that there exists a finite sequence  $(t_0, t_1, \dots, t_n)$  with  $a = t_0 < t_1 < \dots < t_n = b$  and elementary flows  $f_i : [t_{i-1}, t_i] \rightarrow \mathcal{A}$  ( $i = 1, \dots, n$ ) such that

$$\forall s \in [t_{i-1}, t_i]; \quad f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j).$$

The set of all simple (resp. elementary) flows is denoted  $S(\mathcal{A})$  (resp.  $S^e(\mathcal{A})$ ).



According to [24], we use the parametrization of flows which allows to preserve the increment stationarity property under projection on flows (it avoids the appearance of a time-change).

**Definition 4.2.** For any set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  on the space  $(\mathcal{T}, \mathcal{A}, m)$  and any simple flow  $f : [a, b] \rightarrow \mathcal{A}(u)$ , the  $m$ -standard projection of  $X$  on  $f$  is defined as the process

$$X^{f,m} = \left\{ X_t^{f,m} = \Delta X_{f \circ \theta^{-1}(t)}; t \in \theta([a, b]) \right\},$$

where  $\theta$  is the function  $t \mapsto m[f(t)]$  and  $\theta^{-1}$  its right inverse.

In the sequel, we study how regularity of flows connects the exponents  $\alpha_X(U_0)$  (resp.  $\tilde{\alpha}_X(U_0)$ ) and  $\alpha_{X^{f,m}}(t_0)$  (resp.  $\tilde{\alpha}_{X^{f,m}}(t_0)$ ), when  $U_0 \in \mathcal{A}$  and  $f \circ \theta^{-1}(t_0) = U_0$ .

For any  $U_0 \in \mathcal{A}$ , let us denote by  $S(\mathcal{A}, U_0)$  the subset of  $S(\mathcal{A})$  containing all the simple flows  $f : \theta^{-1}(I_f) \rightarrow \mathcal{A}(u)$  such that there exists  $t_0 > 0$  satisfying  $f \circ \theta^{-1}(t_0) = U_0$ , and where  $I_f$  is a closed interval of  $\mathbf{R}_+$  containing a ball centered in  $t_0$ . Such a  $t_0$  does not depend on the flow  $f$ , since  $t_0 = m(U_0)$ . In the same way, we define  $S^e(\mathcal{A}, U_0)$  for elementary flows.

**Lemma 4.3.** Let  $f \in S(\mathcal{A}, U_0)$  and  $\eta > 0$  such that  $B(t_0, \eta) \subset I_f$ . For all  $t \in B(t_0, \eta)$ ,  $f \circ \theta^{-1}(t) \in B_{d_m}^{(u)}(U_0, \eta) = \{A \in \mathcal{A}(u) : m(A \triangle U_0) < \eta\}$ .

*Proof.*  $\theta^{-1}(t) = \inf\{x \in I_f : \theta(x) \geq t\}$ . As  $\theta$  is increasing,  $\theta^{-1}$  is increasing as well. We assume without loss of generality that  $t \geq t_0$ . Then,

$$\begin{aligned} d_m(f \circ \theta^{-1}(t), U_0) &= m(f \circ \theta^{-1}(t) \triangle f \circ \theta^{-1}(t_0)) \\ &= m(f \circ \theta^{-1}(t) \setminus f \circ \theta^{-1}(t_0)) \\ &= m(f \circ \theta^{-1}(t)) - m(f \circ \theta^{-1}(t_0)) \\ &= t - t_0. \end{aligned}$$

□

Using Lemma 4.3, we can compare the Hölder regularity of  $X$  and the Hölder regularity of its projections on flows.

**Proposition 4.4.** Let  $X = \{X_U; U \in \mathcal{A}\}$  be a set-indexed process on  $(\mathcal{T}, \mathcal{A}, m)$ , with finite Hölder exponents at  $U_0 \in \mathcal{A}$ . Then,

$$\begin{aligned} \inf_{f \in S^e(\mathcal{A}, U_0)} \alpha_{X^{f,m}}(t_0) &= \alpha_{X,c}(U_0) \geq \alpha_X(U_0) \quad a.s. \\ \inf_{f \in S^e(\mathcal{A}, U_0)} \tilde{\alpha}_{X^{f,m}}(t_0) &= \tilde{\alpha}_{X,c}(U_0) \geq \tilde{\alpha}_X(U_0) \quad a.s. \end{aligned}$$

where the metric considered on  $\mathcal{A}$  is  $d_m$ .

*Proof.* The proof is only given for the pointwise Hölder exponent. The case of the local Hölder exponent is totally similar.

From Proposition 3.3, the inequality  $\alpha_{X,c}(U_0) \geq \alpha_X(U_0)$  for all  $\omega \in \Omega$  is already known.

The equality  $\inf_{f \in S^e(\mathcal{A}, U_0)} \alpha_{X^{f,m}}(t_0) = \alpha_{X,c}(U_0)$  follows from Corollary 3.5 and Lemma 4.3.

□

The natural question is then to wonder if the previous inequality could be improved in an equality. The answer is generally no, as the following example shows.

**Example 4.5.** *In this example, we only consider deterministic functions, instead of random processes. Let  $F$  be a set-indexed function on  $\mathcal{A}$ , the usual collection of rectangles of  $[0, 1]^2$ . Let  $U_0 \in \mathcal{A}$  and assume that  $F$  is  $\alpha$ -Hölder continuous in  $U_0$ , for some  $\alpha \in (0, 1)$ . We assume without loss of generality that  $F(U_0) = 0$ .*

*Let us divide  $\mathcal{A}$  into four quadrants around  $U_0 = [0, (x_0, y_0)]$  in the following manner:*

$$\begin{aligned} \mathcal{Q}_1 &= \{[0, (x, y)] \in \mathcal{A} : x \leq x_0 \text{ and } y < y_0\}, \\ \mathcal{Q}_2 &= \{[0, (x, y)] \in \mathcal{A} : x \leq x_0 \text{ and } y \geq y_0\}, \\ \mathcal{Q}_3 &= \{[0, (x, y)] \in \mathcal{A} : x > x_0 \text{ and } y \geq y_0\}, \\ \mathcal{Q}_4 &= \{[0, (x, y)] \in \mathcal{A} : x > x_0 \text{ and } y < y_0\}. \end{aligned}$$

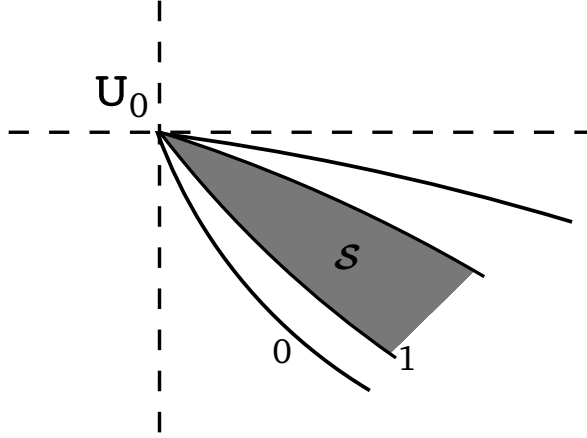


FIGURE 2. Value of  $G$  around  $U_0$

*Let us fix  $\epsilon > 0$ . As  $F$  is  $\alpha$ -Hölder continuous at  $U_0$ , for all  $K > 0$ , there exists a sequence of sets in  $\mathcal{A}$  converging to  $U_0$  and such that*

$$\forall n \geq 0, \quad |F(U_n)| = |F(U_n) - F(U_0)| > K d_{\mathcal{A}}(U_n, U_0)^{\alpha+\epsilon}.$$

*There is at least one of the quadrants in which there are infinitely many sets  $U_n$ . Up to a rotation, assume  $\mathcal{Q}_4$  is this quadrant. We now assume (without restriction) that a subsequence of  $(U_n)$  belongs to a closed subset  $\mathcal{S} \subset \mathcal{Q}_4$  (see figure 2).*

*Let  $G$  be a smooth function except maybe at  $U_0$ , taking its values in  $[0, 1]$  and such that  $G(U_0) = 0$  and  $G(U) = 0$  for all  $U \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$  (see figure 2 above), and  $G(U) = 1$  for all  $U \in \mathcal{S}$ . We denote by  $H$  the product of  $F$  and  $G$ .*

*Up to an extraction that we detailed previously, the sequence  $(U_n)_{n \in \mathbb{N}}$  belongs to  $\mathcal{S}$ . Then,*

$$\forall n \geq 0, \quad |H(U_n) - H(U_0)| = |H(U_n)| = |F(U_n)| > K d_{\mathcal{A}}(U_n, U_0)^{\alpha+\epsilon}.$$

*Thus, if  $H$  is  $\beta$ -Hölder continuous at  $U_0$ , then the inequality  $\beta \leq \alpha$  holds necessarily.*

For  $\gamma < \alpha$ , there exist  $\rho > 0$  and  $K > 0$  such that

$$\forall U \in B_{d_A}(U_0, \rho), \quad |F(U)| = |F(U) - F(U_0)| \leq K d_A(U, U_0)^\gamma.$$

Thus,

$$|H(U) - H(U_0)| = |H(U)| \leq G(U) \cdot |F(U)| \leq K d_A(U, U_0)^\gamma.$$

We have built a function  $H$  which is  $\alpha$ -Hölder continuous. On the other hand, the projection of  $H$  on any elementary flow  $f \in S^e(\mathcal{A}, U_0)$  is uniformly 0 and consequently,  $\inf_{f \in S^e(\mathcal{A}, U_0)} \tilde{\alpha}_{Hf, m} = \infty > \alpha$ .

## 5. ALMOST SURE VALUES FOR THE HÖLDER EXPONENTS

As in the real-parameter case, we prove that the random Hölder exponents of the paths have almost sure values, when the process is Gaussian.

Defining Hölder exponents by (3.3) and (3.4) leads us to ask whether they are random variables, in order to consider measurable events related to these quantities. This question was first answered by Doob (see [17]) for linear parameter space. The definition for set-indexed processes reduces to real-parameter processes when the indexing collection is the rectangles of  $\mathbf{R}_+$ . To our knowledge, the question of the existence of a separable modification of a set-indexed process has not been answered so far. Instead, Skorokhod topologies have been adapted to the set-indexed framework (see [13, 3]) giving that objects such as  $\sup_{U \in \mathcal{A}} X_U$  are measurable for suitable processes (ie outer continuous with inner limit).

**Definition 5.1.** A process  $\{X_U, U \in \mathcal{A}\}$  is said separable if there exist an at most countable collection  $\mathcal{S} \subset \mathcal{A}$  and a null set  $\Lambda$  such that for all closed sets  $F \subset \mathbf{R}$  and all open set  $\mathcal{O}$  for the topology induced by  $d_A$ ,

$$\{\omega : X_U(\omega) \in F \text{ for all } U \in \mathcal{O} \cap \mathcal{S}\} \setminus \{\omega : X_U(\omega) \in F \text{ for all } U \in \mathcal{O}\} \subset \Lambda$$

Each point of the original proof of Doob (we refer to [17, 28]) can be extended to the set-indexed setting. We can state:

**Theorem 5.2** (Doob's separability Theorem, set-indexed framework). *If the subcollections  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  and the metric  $d_A$  satisfy Assumption 1, any set-indexed stochastic process  $X = \{X_U; U \in \mathcal{A}\}$  has a separable modification.*

We shall now consider that all our processes are separable. As a consequence, assuming without any restriction on the probability space, variables such as  $\sup_{U \in \mathcal{O}} X_U$ , for  $\mathcal{O}$  an open set of  $\mathcal{A}$ , are indeed measurable. Hence the random Hölder coefficients aforementioned are random variables.

Given those considerations, we can turn to the two main results of this section. The following assumption will be required to prove the almost sure value of the pointwise Hölder exponent.

**Assumption 2.** *The subclasses  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  of the indexing collection  $\mathcal{A}$  satisfy the two following assertions:*

(1) *For all integer  $\kappa \geq 0$  and for all real  $\beta > 0$ ,*

$$\sum_{n=0}^{+\infty} \frac{k_{(1+\kappa)n}}{\exp k_n^\beta} < +\infty.$$

(2) For all integer  $\kappa \geq 0$ ,

$$\forall n \in \mathbf{N}, \quad k_{(1+\kappa)n} \geq k_n^\kappa.$$

A priori, the two conditions of Assumption 2 may seem incompatible since Condition (1) implies that  $k_n$  does not grow too fast as  $n$  goes to  $\infty$ , and Condition (2) implies that  $k_n$  grows sufficiently fast. However we can observe that any sequence  $(k_n)_{n \in \mathbf{N}}$  such that  $k_n = e^{an}$  for all  $n \in \mathbf{N}$ , where  $a \in \mathbf{R}_+$ , satisfies both Conditions (1) and (2). In particular, the sequence  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  of dyadic rectangles of  $\mathbf{R}_+^N$  satisfy Assumption 2. There are also sequences with much faster growth rate satisfying these conditions (think of  $(\exp\{\exp(\dots \exp(n))\})_n$ ).

Recall that according to Remark 2.11, Condition (H2) can be removed from Assumption 1 when the process  $X$  is Gaussian and therefore in all this section.

Let us define the *deterministic pointwise Hölder exponent*

$$\alpha_X(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{U, V \in B_{d_A}(U_0, \rho)} \frac{\mathbf{E}[|X_U - X_V|^2]}{\rho^{2\alpha}} < \infty \right\} \quad (5.1)$$

and the *deterministic local Hölder exponent*

$$\tilde{\alpha}_X(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{U, V \in B_{d_A}(U_0, \rho)} \frac{\mathbf{E}[|X_U - X_V|^2]}{d_A(U, V)^{2\alpha}} < \infty \right\}. \quad (5.2)$$

These exponents satisfy

$$\forall U_0 \in \mathcal{A}; \quad \tilde{\alpha}_X(U_0) \leq \alpha_X(U_0).$$

**Theorem 5.3.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  a set-indexed centered Gaussian process, where  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  and  $d_A$  satisfy Assumption 1. If the deterministic local Hölder exponent of  $X$  at  $U_0 \in \mathcal{A}$  is positive and finite, we have*

$$\mathbf{P} \{ \tilde{\alpha}_X(U_0) = \tilde{\alpha}_X(U_0) \} = 1.$$

Moreover, if Assumption 2 also holds, then

$$\mathbf{P} \{ \alpha_X(U_0) = \alpha_X(U_0) \} = 1.$$

In a similar way to Theorem 3.14 of [22], we can also obtain almost sure results on the exponents  $\alpha_X(U_0)$  and  $\tilde{\alpha}_X(U_0)$  uniformly in  $U_0 \in \mathcal{A}$ .

**Theorem 5.4.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be a set-indexed centered Gaussian process, where  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  and  $d_A$  satisfy Assumption 1.*

*Suppose that the functions  $U_0 \mapsto \liminf_{U \rightarrow U_0} \tilde{\alpha}_X(U)$  and  $U_0 \mapsto \liminf_{U \rightarrow U_0} \alpha_X(U)$  are positive over  $\mathcal{A}$ . Then, with probability one,*

$$\forall U_0 \in \mathcal{A}, \quad \liminf_{U \rightarrow U_0} \tilde{\alpha}_X(U) \leq \tilde{\alpha}_X(U_0) \leq \limsup_{U \rightarrow U_0} \tilde{\alpha}_X(U) \quad (5.3)$$

and under Assumption 2,

$$\forall U_0 \in \mathcal{A}, \quad \liminf_{U \rightarrow U_0} \alpha_X(U) \leq \alpha_X(U_0). \quad (5.4)$$

The proof of Theorem 5.3 is an adaptation of proofs in [20]. For sake of completeness, we detail them in the two following sections. The proof of Theorem 5.4 is given in Section 5.3.

**5.1. Lower bound for the pointwise and local Hölder exponents.** A lower bound for the local Hölder exponent is directly given by Corollary 2.10.

For all  $U_0 \in \mathcal{A}$  and all  $0 < \alpha < \tilde{\omega}_X(U_0)$ , there exists  $\rho_0 > 0$  and  $K > 0$  such that

$$\forall U, V \in B_{d_{\mathcal{A}}}(U_0, \rho_0); \quad \mathbf{E} [|X_U - X_V|^2] \leq K d_{\mathcal{A}}(U, V)^{2\alpha}.$$

Therefore, the sample path of  $X$  is almost surely  $\nu$ -Hölder continuous in  $B_{d_{\mathcal{A}}}(U_0, \rho_0)$  for all  $\nu \in (0, \alpha)$ , which leads to  $\alpha \leq \tilde{\alpha}_X(U_0)$  almost surely. Then we get

$$\mathbf{P} \{ \tilde{\alpha}_X(U_0) \geq \tilde{\omega}_X(U_0) \} = 1.$$

By inequality (3.5), any lower bound for the local Hölder exponent is also a lower bound for the pointwise exponent. Moreover it can be improved in the case of strict inequality  $0 < \tilde{\omega}_X(U_0) < \omega_X(U_0)$ .

As previously, for all  $U_0 \in \mathcal{A}$ , there exist  $\alpha > 0$ ,  $\rho_0 > 0$  and a modification of  $X$ , which is  $\nu$ -Hölder continuous in  $B_{d_{\mathcal{A}}}(U_0, \rho_0)$  for all  $\nu \in (0, \alpha)$ , i.e. there exists a random variable  $h^*$  and a constant  $L > 0$  such that almost surely

$$\forall U, V \in B_{d_{\mathcal{A}}}(U_0, \rho_0); \quad d_{\mathcal{A}}(U, V) < h^* \Rightarrow |X_U - X_V| \leq L d_{\mathcal{A}}(U, V)^\nu.$$

In the following, we consider such a  $\nu$  with  $\frac{1}{\nu} \in \mathbf{N}$ .

For any  $\epsilon > 0$ , there exist  $0 < \rho_1 < \rho_0$  and  $M > 0$  such that

$$\forall \rho < \rho_1, \quad \forall U, V \in B(U_0, \rho); \quad \mathbf{E} \left[ \left| \frac{X_U - X_V}{\rho^{\omega_X(U_0) - \epsilon}} \right|^2 \right] \leq M \rho^\epsilon.$$

Then setting  $\gamma = \omega_X(U_0) - \epsilon$ , the exponential inequality for the centered Gaussian variable  $X_U - X_V$  implies

$$\mathbf{P} \{ |X_U - X_V| \geq \rho^\gamma \} \leq \exp \left( -\frac{1}{2} \frac{\rho^{2\gamma}}{\mathbf{E} [|X_U - X_V|^2]} \right) \leq \exp \left( -\frac{1}{2} M \rho^\epsilon \right).$$

We consider the particular case  $\rho = M_1 k_n^{-1/q_{\mathcal{A}}} < \rho_1$  for  $n \in \mathbf{N}$  large enough, and for any  $i \in \mathbf{N}$ ,  $D_i = B(U_0, M_1 k_n^{-1/q_{\mathcal{A}}}) \cap \mathcal{A}_{n+i}$ .

Then, since  $\#D_i \leq k_{n+i}$ , and taking  $i$  to be  $\frac{1+|\gamma|}{\nu} n = \kappa n$ ,

$$\mathbf{P} \left( \sup_{U \in D_{\kappa n}} |X_U - X_{U_0}| \geq M_1^\gamma k_n^{-\gamma/q_{\mathcal{A}}} \right) \leq k_{(1+\kappa)n} \exp \left( -\frac{1}{2} M M_1^\epsilon k_n^{-\epsilon/q_{\mathcal{A}}} \right).$$

Using Condition (1) of Assumption 2, Borel-Cantelli Lemma implies the existence of a random variable  $n^{**}$  such that

$$\forall n \geq n^{**}, \quad \sup_{U \in D_{\kappa n}} |X_U - X_{U_0}| < M_1^\gamma k_n^{-\gamma/q_{\mathcal{A}}} \quad \text{a.s.} \quad (5.5)$$

Let us fix any integer  $N \geq n^{**}$  such that  $M_1 k_N^{-1/q_{\mathcal{A}}} < h^*$  and consider any  $U \in B(U_0, M_1 k_N^{-1/q_{\mathcal{A}}}) \cap \mathcal{D}$ , where  $\mathcal{D} = \bigcup_{n \in \mathbf{N}} \mathcal{A}_n$ .

- If  $U \in \mathcal{A}_{(1+\kappa)N}$  then  $|X_U - X_{U_0}| < M_1^\gamma k_N^{-\gamma/q_{\mathcal{A}}}$  a.s. by (5.5).

- Otherwise, let us consider  $U' = g_{(1+\kappa)N}(U)$ . We have  $d_{\mathcal{A}}(U, U') \leq M_1 k_{(1+\kappa)N}^{-1/q_{\mathcal{A}}}$  and then,  $U' \in B(U_0, 2M_1 k_N^{-1/q_{\mathcal{A}}})$  by the triangular inequality. Let us remark that using the same method, statement (5.5) can be easily improved in

$$\forall n \geq n^{**}, \quad \sup_{U \in B(U_0, 2M_1 k_n^{-1/q_{\mathcal{A}}}) \cap \mathcal{A}_{(1+\kappa)n}} |X_U - X_{U_0}| < M_1^\gamma k_n^{-\gamma/q_{\mathcal{A}}} \quad \text{a.s.}$$

Consequently, using Condition (2) of Assumption 2,

$$\begin{aligned} |X_U - X_{U_0}| &\leq |X_U - X_{U'}| + |X_{U'} - X_{U_0}| \\ &\leq L d_{\mathcal{A}}(U, U')^\nu + M_1^\gamma k_N^{-\gamma/q_{\mathcal{A}}} \\ &\leq L M_1^\nu k_{(1+\kappa)N}^{-\nu/q_{\mathcal{A}}} + M_1^\gamma k_N^{-\gamma/q_{\mathcal{A}}} \leq L' k_N^{-\gamma/q_{\mathcal{A}}} \end{aligned} \quad (5.6)$$

for some constant  $L' > 0$ .

This inequality leads to

$$\sup_{U, V \in B(U_0, M_1 k_N^{-1/q_{\mathcal{A}}})} |X_U - X_V| \leq C k_N^{-\gamma/q_{\mathcal{A}}} \quad \text{a.s.}$$

and since the sequence  $\left(M_1 k_n^{-1/q_{\mathcal{A}}}\right)_{n \in \mathbf{N}}$  is non-increasing,

$$\limsup_{\rho \rightarrow 0} \sup_{U, V \in B(U_0, \rho)} \frac{|X_U - X_V|}{\rho^\gamma} < \infty \quad \text{a.s.}$$

Therefore,  $\forall \epsilon > 0$ ,  $\alpha_X(U_0) \geq \mathfrak{o}_X(U_0) - \epsilon$  almost surely and  $\mathbf{P}(\alpha_X(U_0) \geq \mathfrak{o}_X(U_0)) = 1$ .

**5.2. Upper bounds for the pointwise and local Hölder exponents.** As in [20], upper bounds for the pointwise and local Hölder exponents are given by the following two lemmas. Their proof are totally identical to multiparameter setting.

**Lemma 5.5.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be a centered Gaussian process. Assume that for  $U_0 \in \mathcal{A}$ , there exists  $\mu \in (0, 1)$  such that for all  $\epsilon > 0$ , there exist a sequence  $(U_n)_{n \in \mathbf{N}^*}$  of  $\mathcal{A}$  converging to  $U_0$ , and a constant  $c > 0$  such that*

$$\forall n \in \mathbf{N}^*; \quad \mathbf{E}[|X_{U_n} - X_{U_0}|^2] \geq c d_{\mathcal{A}}(U_n, U_0)^{2\mu+\epsilon}.$$

*Then, we have almost surely*

$$\alpha_X(U_0) \leq \mu.$$

Since the process  $X$  has a finite deterministic Hölder exponent, for  $\mu = \mathfrak{o}_X(U_0)$ , one can find a sequence  $(U_n)$  as in Lemma 5.5. Hence  $\mathbf{P}(\alpha_X(U_0) \leq \mathfrak{o}_X(U_0)) = 1$ .

**Lemma 5.6.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be a centered Gaussian process. Assume that for  $U_0 \in \mathcal{A}$ , there exists  $\mu \in (0, 1)$  such that for all  $\epsilon > 0$ , there exist two sequences  $(U_n)_{n \in \mathbf{N}^*}$  and  $(V_n)_{n \in \mathbf{N}^*}$  of  $\mathcal{A}$  converging to  $U_0$ , and a constant  $c > 0$  such that*

$$\forall n \in \mathbf{N}^*; \quad \mathbf{E}[|X_{U_n} - X_{V_n}|^2] \geq c d_{\mathcal{A}}(U_n, V_n)^{2\mu+\epsilon}.$$

*Then, we have almost surely*

$$\tilde{\alpha}_X(U_0) \leq \mu.$$

As for the pointwise case,  $\mathbf{P}(\tilde{\alpha}_X(U_0) \leq \tilde{\omega}_X(U_0)) = 1$  follows from Lemma 5.6 with  $\mu = \tilde{\omega}_X(U_0)$ .

**5.3. Proof of the uniform almost sure result.** This section is devoted to the proof of Theorem 5.4. We only consider the local Hölder exponent. The uniform almost sure lower bound for the pointwise exponent is proved in a similar way.

Starting with the lower bound, from Theorem 2.9, for all  $U_0 \in \mathcal{A}$  and all  $\epsilon > 0$ , there is a modification  $Y_{U_0}$  of  $X$  which is  $\alpha$ -Hölder continuous for all  $\alpha \in (0, \tilde{\omega}_X(U_0) - \epsilon)$  on  $B_{d_A}(U_0, \rho_0)$ .

- In the first step,  $\tilde{\omega}_X$  is assumed to be constant over  $\mathcal{A}$ . Hence the local Hölder exponent of  $Y_{U_0}$  satisfies almost surely

$$\forall U \in B_{d_A}(U_0, \rho_0), \quad \tilde{\alpha}_{Y_{U_0}}(U) \geq \tilde{\omega}_X - \epsilon. \quad (5.7)$$

The collection  $\mathcal{A}$  is totally bounded, so it can be covered by a countable number of balls of radius at most  $\eta$ , for all  $\eta > 0$ . Let  $B$  be one of these balls. For all  $U_0 \in \mathcal{A}$ , we consider  $\rho_0 > 0$  such that (5.7) holds. We have obviously

$$B \subseteq \bigcup_{U_0 \in B} B_{d_A}(U_0, \rho_0).$$

For each open ball, there exists an integer  $n$  such that  $B_{d_A}(U_0, \rho_0) \cap \mathcal{A}_n \neq \emptyset$  so that for  $V_0 \in B_{d_A}(U_0, \rho_0) \cap \mathcal{A}_n$ , there exists an integer  $m_0$  such that  $U_0 \in B_{d_A}(V_0, 2^{-m_0}) \subseteq B_{d_A}(U_0, \rho_0)$ . Thus

$$B \subseteq \bigcup B_{d_A}(V_0, 2^{-m_0}),$$

where the union is countable. Each of these balls satisfies

$$\mathbf{P}(\forall U \in B_{d_A}(V_0, 2^{-m_0}), \tilde{\alpha}_X(U) \geq \tilde{\omega}_X - \epsilon) = 1,$$

and since  $\mathcal{A}$  is a countable union of balls  $B_{d_A}(V_0, 2^{-m_0})$ , we get

$$\mathbf{P}(\forall U \in \mathcal{A}, \tilde{\alpha}_X(U) \geq \tilde{\omega}_X - \epsilon) = 1.$$

Taking  $\epsilon \in \mathbf{Q}_+^*$ , we conclude that

$$\mathbf{P}(\forall U \in \mathcal{A}, \tilde{\alpha}_X(U) \geq \tilde{\omega}_X) = 1. \quad (5.8)$$

- In the general case of a not constant exponent  $\tilde{\omega}_X$ , for any ball  $B$  of radius  $\eta$  previously introduced, we set  $\beta = \inf_{U \in B} \tilde{\omega}_X(U) - \epsilon$ ,  $\epsilon > 0$ . Then, there exists a constant  $C > 0$  such that

$$\forall U, V \in B, \quad \mathbf{E}[|X_U - X_V|^2] \leq C d_A(U, V)^{2\beta}.$$

In a similar way as we proved (5.8), we deduce the existence of an event  $\Omega^* \subseteq \Omega$  of probability one such that for all  $\omega \in \Omega^*$ :

$$\forall U \in \mathcal{A}, \forall n \geq 0, \forall \epsilon \in \mathbf{Q}_+^*,$$

$$\forall U_0 \in B_{d_A}(U, 2^{-n}), \quad \tilde{\alpha}_X(U_0) \geq \inf_{V \in B_{d_A}(U, 2^{-n})} \tilde{\omega}_X(V) - \epsilon.$$

By letting  $n \rightarrow \infty$ , the previous equation leads to

$$\mathbf{P}\left(\forall U_0 \in \mathcal{A}, \quad \tilde{\alpha}_X(U_0) \geq \liminf_{U \rightarrow U_0} \tilde{\omega}_X(U)\right) = 1.$$



In order to prove the converse inequality (which holds only for the local exponent), we adapt a proof in [22]. We first assume that  $\tilde{\alpha}_X$  is constant on  $\mathcal{A}$ .

Using the fact that  $\mathcal{D} = \bigcup_{n \in \mathbf{N}} \mathcal{A}_n$  is countable, Lemma 5.6 gives

$$\mathbf{P}(\forall U \in \mathcal{D}, \tilde{\alpha}_X(U) \leq \tilde{\alpha}_X) = 1.$$

Let  $\Omega' \in \mathcal{F}$  be the set of  $\omega$ , such that  $\tilde{\alpha}_X(U) \leq \tilde{\alpha}_X$  for all  $U \in \mathcal{D}$ . Let  $U_0 \in \mathcal{A} \setminus \mathcal{D}$ . Let  $(U^{(i)})_{i \in \mathbf{N}}$  be a sequence in  $\mathcal{D}$  converging to  $U_0$ . On  $\Omega'$ ,  $\tilde{\alpha}_X(U^{(i)}) \leq \tilde{\alpha}_X$ , for all  $i \in \mathbf{N}$ . For each fixed  $i \in \mathbf{N}$ , there exist two sequences  $(V_n^{(i)})_{n \in \mathbf{N}}$  and  $(W_n^{(i)})_{n \in \mathbf{N}}$  in  $\mathcal{A}$  converging to  $U^{(i)}$  as  $n \rightarrow \infty$ , and for all  $n \in \mathbf{N}$ ,

$$\lim_{n \rightarrow +\infty} \frac{|X_{V_n^{(i)}} - X_{W_n^{(i)}}|}{d_{\mathcal{A}}(V_n^{(i)}, W_n^{(i)})^{\tilde{\alpha}_X + \epsilon}} = +\infty.$$

As in [22], we build two other sequences  $(V_n)_{n \in \mathbf{N}}$  and  $(W_n)_{n \in \mathbf{N}}$  so that  $V_n \rightarrow U_0$  and  $W_n \rightarrow U_0$  and

$$\lim_{n \rightarrow +\infty} \frac{|X_{V_n} - X_{W_n}|}{d_{\mathcal{A}}(V_n, W_n)^{\tilde{\alpha}_X + \epsilon}} = +\infty.$$

This implies the expected inequality for all  $U_0 \in \mathcal{A}$ .

The general case for  $\tilde{\alpha}_X$  not constant is proved in the same way as for the lower bound.

**5.4. Corollaries for the  $\mathcal{C}$ -Hölder exponents and the pointwise continuity exponent.** Theorem 5.3 can be transposed to the  $\mathcal{C}$ -Hölder exponent, and the pointwise continuity exponent.

If  $X$  is a Gaussian set-indexed process, we define respectively the deterministic pointwise and local  $\mathcal{C}$ -Hölder exponents on one hand, for all  $l \in \mathbf{N}^*$ :

$$\begin{aligned} \alpha_{X, \mathcal{C}^l}(U_0) &= \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ V \in \mathcal{B}^l(U_0, \rho)}} \frac{\mathbf{E}[|\Delta X_{U \setminus V}|^2]}{\rho^{2\alpha}} < \infty \right\}, \\ \tilde{\alpha}_{X, \mathcal{C}^l}(U_0) &= \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ V \in \mathcal{B}^l(U_0, \rho)}} \frac{\mathbf{E}[|\Delta X_{U \setminus V}|^2]}{d_{\mathcal{A}}(U, V)^{2\alpha}} < \infty \right\}, \end{aligned}$$

and the deterministic pointwise continuity exponent on the other hand,

$$\alpha_X^{pc}(t_0) = \sup \left\{ \alpha : \limsup_{n \rightarrow \infty} \frac{\mathbf{E}[|\Delta X_{C_n(t)}|^2]}{m(C_n(t))^{2\alpha}} < \infty \right\}.$$

Similarly to Proposition 3.3, the pointwise and local deterministic exponents do not depend on  $l$ . Hence they are denoted respectively  $\alpha_{X, \mathcal{C}}(U_0)$  and  $\tilde{\alpha}_{X, \mathcal{C}}(U_0)$ .

**Corollary 5.7.** *Let  $X = \{X_U, U \in \mathcal{A}\}$  be a centered Gaussian set-indexed process. If the subcollections  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  satisfy Assumption 1 and if the deterministic  $\mathcal{C}$ -Hölder exponents are finite, then for  $U_0 \in \mathcal{A}$ ,*

$$\alpha_{X, \mathcal{C}}(U_0) = \alpha_{X, \mathcal{C}}(U_0) \text{ a.s.} \quad \text{and} \quad \tilde{\alpha}_{X, \mathcal{C}}(U_0) = \tilde{\alpha}_{X, \mathcal{C}}(U_0) \text{ a.s.}$$

*Proof.* It suffices to prove the result for  $l = 1$ , which corresponds to  $V \subseteq U$  in the definition of the standard Hölder exponent. Thus one can apply the previous proofs (Sections 5.1 and 5.2) which are still valid when restricted to  $V \subseteq U$ .  $\square$

**Corollary 5.8.** *Let  $X = \{X_U, U \in \mathcal{A}\}$  be a centered Gaussian set-indexed process. If the deterministic exponent of pointwise continuity is finite, then for  $t_0 \in \mathcal{T}$ ,*

$$\alpha_X^{pc}(t_0) = \mathfrak{a}_X^{pc}(t_0) \quad \text{a.s.}$$

Moreover, for any  $U_{max} \in \mathcal{A}$  such that  $m(U_{max}) < \infty$ ,

$$\mathbf{P}(\forall t \in U_{max}, \alpha_X^{pc}(t) \geq \mathfrak{a}_X^{pc}(t)) = 1.$$

*Proof.* Fix  $t_0 \in \mathcal{T}$ . Let  $\alpha < \mathfrak{a}_X^{pc}(t_0)$ . The inequality  $\alpha < \alpha_X^{pc}(t_0)$  a.s. is a straightforward consequence of Proposition 3.8. This gives  $\alpha_X^{pc}(t_0) \geq \mathfrak{a}_X^{pc}(t_0)$  almost surely.

For the converse inequality, denote  $\mu = \mathfrak{a}_X^{pc}(t_0)$ . Then for all  $\epsilon > 0$ , there exists a subsequence  $(C_{\varphi(n)}(t_0))_{n \in \mathbf{N}}$  of  $(C_n(t_0))_{n \in \mathbf{N}}$  and a constant  $c > 0$  such that

$$\forall n \in \mathbf{N}^*, \quad \mathbf{E}[|\Delta X_{C_{\varphi(n)}(t_0)}|^2] \geq c m(C_{\varphi(n)}(t_0))^{2\mu+\epsilon}.$$

For all  $n \in \mathbf{N}$ , the law of the random variable  $\frac{\Delta X_{C_{\varphi(n)}(t_0)}}{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}}$  is  $\mathcal{N}(0, \sigma_n^2)$ . The previous inequality implies that  $\sigma_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then for all  $\lambda > 0$ , the same computation as in Lemmas 5.5 and 5.6 leads to

$$\begin{aligned} \mathbf{P}\left(\frac{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}}{\Delta X_{C_{\varphi(n)}(t_0)}} < \lambda\right) &= \mathbf{P}\left(\frac{\Delta X_{C_{\varphi(n)}(t_0)}}{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}} > \frac{1}{\lambda}\right) \\ &= \int_{|x| > \frac{1}{\lambda}} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{x^2}{2\sigma_n^2}\right) dx \\ &= \frac{1}{2\pi} \int_{|x| > \frac{1}{\lambda\sigma_n}} \exp\left(-\frac{x^2}{2}\right) dx \xrightarrow{n \rightarrow +\infty} 1. \end{aligned}$$

Therefore the sequence  $\left(\frac{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}}{\Delta X_{C_{\varphi(n)}(t_0)}}\right)_{n \in \mathbf{N}}$  converges to 0 in probability. then there exists a subsequence which converges to 0 almost surely. Then for all  $\epsilon > 0$ , we have almost surely  $\alpha_X^{pc}(t_0) \leq \mu + \epsilon$ . Taking  $\epsilon \in \mathbf{Q}_+$ , this yields  $\alpha_X^{pc}(t_0) \leq \mathfrak{a}_X^{pc}(t_0)$  a.s.

The second equation is a straightforward consequence of Proposition 3.9.  $\square$

## 6. APPLICATION: HÖLDER REGULARITY OF THE SET-INDEXED FRACTIONAL BROWNIAN MOTION AND OF THE SET-INDEXED ORNSTEIN-ÜLHENBECK PROCESS

**6.1. Hölder exponents of the SIfBm.** In the case of one-parameter fractional Brownian motion  $B^H$ , the local regularity of the sample paths is given by the self-similarity index  $H \in (0, 1)$ . More precisely, the two classical Hölder exponents satisfy, with probability one,

$$\forall t \in \mathbf{R}_+; \quad \alpha_{B^H}(t) = \tilde{\alpha}_{B^H}(t) = H.$$

In [23, 24], a set-indexed extension for fractional Brownian motion has been defined and studied. A mean-zero Gaussian process  $\mathbf{B}^H = \{\mathbf{B}_U^H, U \in \mathcal{A}\}$  is called a *set-indexed fractional Brownian motion (SIfBm for short)* on  $(\mathcal{T}, \mathcal{A}, m)$  if

$$\forall U, V \in \mathcal{A}, \quad \mathbf{E} [\mathbf{B}_U^H \mathbf{B}_V^H] = \frac{1}{2} [m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}], \quad (6.1)$$

where  $H \in (0, 1/2]$  is the index of self-similarity of the process.

In [21], the deterministic local Hölder exponent and the local Hölder exponent have been determined for the particular case of an SIfBm indexed by the collection  $\{[0, t]; t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$ , called the *multiparameter fractional Brownian motion*. If  $X$  denotes the  $\mathbf{R}_+^N$ -indexed process defined by  $X_t = \mathbf{B}_{[0,t]}^H$  for all  $t \in \mathbf{R}_+^N$ , it is proved that for all  $t_0 \in \mathbf{R}_+^N$ ,  $\tilde{\alpha}_X(t_0) = H$  and with probability one, for all  $t_0 \in \mathbf{R}_+^N$ ,  $\tilde{\alpha}_X(t_0) = H$ . Theorem 5.3 allows to extend these results to SIfBm indexed by a more general class than the sole collection of rectangles of  $\mathbf{R}_+^N$ .

In Section 5, Theorem 5.4 failed to provide a uniform almost sure upper bound for the pointwise Hölder exponent of a general Gaussian set-indexed process. In the specific case of the set-indexed fractional Brownian motion, this result can be improved under some additional requirement. In that view, we consider a supplementary condition on the collection  $\mathcal{A}$  and the distance  $d_m$ : there exists  $\eta > 0$  such that  $\forall U_0 \in \mathcal{A}$ ,

$$\inf_{\rho \rightarrow 0} \sup \left\{ \frac{d_m(U, g_n(U))}{\rho}; n \in \mathbf{N}, U, g_n(U) \in B_{d_m}(U_0, \rho) \right\} \geq \eta. \quad (6.2)$$

**Theorem 6.1.** *Let  $\mathbf{B}^H$  be a set-indexed fractional Brownian motion on  $(\mathcal{T}, \mathcal{A}, m)$ ,  $H \in (0, 1/2]$ . Assume that the subclasses  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  satisfy Assumption 1. Then, the local and pointwise Hölder exponents of  $\mathbf{B}^H$  at any  $U_0 \in \mathcal{A}$ , defined with respect to the distance  $d_m$  or any equivalent distance, satisfy*

$$\mathbf{P} (\forall U_0 \in \mathcal{A}, \tilde{\alpha}_{\mathbf{B}^H}(U_0) = H) = 1$$

and if Assumption 2 and the additional Condition (6.2) hold,

$$\mathbf{P} (\forall U_0 \in \mathcal{A}, \alpha_{\mathbf{B}^H}(U_0) = H) = 1.$$

Consequently, when  $\mathcal{A}$  is the collection of rectangles of  $\mathbf{R}_+^N$  and  $m$  is the Lebesgue measure, i.e.  $\mathbf{B}^H$  is a multiparameter fractional Brownian motion, we have

$$\mathbf{P} (\forall U_0 \in \mathcal{A}, \alpha_{\mathbf{B}^H}(U_0) = \tilde{\alpha}_{\mathbf{B}^H}(U_0) = H) = 1.$$

*Proof.* From the definition of the set-indexed fractional Brownian motion, the following expression of the incremental variance,

$$\forall U, V \in \mathcal{A}, \quad \mathbf{E} [|\mathbf{B}_U^H - \mathbf{B}_V^H|^2] = m(U \triangle V)^{2H},$$

directly implies that the deterministic pointwise and local Hölder exponents are equal to  $H$ . By Theorem 5.3, the random exponents on an indexing collection satisfying Assumptions 1 and 2 are also equal to  $H$ .

For the uniform almost sure result on  $\mathcal{A}$ , according Theorem 5.4, it remains to prove that  $\mathbf{P} (\forall U_0 \in \mathcal{A}, \alpha_{\mathbf{B}^H}(U_0) \leq H) = 1$ . This fact is the object of the following Section 6.2.

For the particular case of the multiparameter fractional Brownian motion, it suffices to notice that the collection  $\mathcal{A}$  of rectangles of  $\mathbf{R}^N$  endowed with the Lebesgue measure  $\lambda$  satisfies Condition (6.2).

Let us recall that for any  $U_0 \in \mathcal{A}$ ,  $d_\lambda(U_0, g_n(U_0)) = N \cdot 2^{-n} + o(2^{-n})$ . Hence for a given  $\rho > 0$ , choosing the smallest integer  $n$  such that  $N \cdot 2^{-n} \leq \rho/2$  ensures that

$$\frac{d_\lambda(U_0, g_n(U_0))}{\rho} \geq \frac{N \cdot 2^{-(n+1)}}{\rho} \geq \frac{1}{8},$$

and that  $g_n(U_0) \in B_{d_\lambda}(U_0, \rho)$ . □

If the collection  $\mathcal{A}$  or the metric  $d_m$  do not satisfy the additional requirement (6.2), then the lower bound for the pointwise exponent remains true by Theorem 5.4:  $\mathbf{P}(\forall U_0 \in \mathcal{A}, \alpha_{\mathbf{B}^H}(U_0) \geq H) = 1$ .

In [23], it is shown that for all  $U, V \in \mathcal{A}$ ,  $\mathbf{E}[|\Delta \mathbf{B}_{U \setminus V}^H|^2] = m(U \setminus V)^{2H}$ . This implies that for all  $U_0 \in \mathcal{A}$ ,  $\tilde{\omega}_{\mathbf{B}^H, \mathcal{C}}(U_0) = \omega_{\mathbf{B}^H, \mathcal{C}}(U_0) = H$ , and so by Corollary 5.7:

$$\tilde{\alpha}_{\mathbf{B}^H, \mathcal{C}}(U_0) = \alpha_{\mathbf{B}^H, \mathcal{C}}(U_0) = H \quad \text{a.s.}$$

The case of the exponent of pointwise continuity needs to determine the behaviour of  $\mathbf{E}[|\Delta \mathbf{B}_C^H|^2]$  when  $C \in \mathcal{C}$  (and not only  $C = U \setminus V \in \mathcal{C}_0$ , with  $U, V \in \mathcal{A}$  as previously). In the specific case of an SIfBm with  $H = 1/2$ , we can state:

**Proposition 6.2.** *Let  $\mathbf{B}$  be a Brownian motion on  $\mathcal{A}$ . Then, for all  $t_0 \in \mathcal{T}$ ,*

$$\alpha_{\mathbf{B}}^{pc}(t_0) = \omega_{\mathbf{B}}^{pc}(t_0) = \frac{1}{2} \quad \text{a.s.}$$

*A uniform lower bound in any  $U_{max} \in \mathcal{A}$  such that  $m(U_{max}) < \infty$ , is given by:*

$$\mathbf{P}\left(\forall t_0 \in U_{max}, \quad \alpha_{\mathbf{B}}^{pc}(t_0) \geq \omega_{\mathbf{B}}^{pc}(t_0) = \frac{1}{2}\right) = 1.$$

*Proof.* It suffices to notice that  $\mathbf{E}[|\Delta \mathbf{B}_C|^2] = m(C)$  and this is a consequence of Corollary 5.8. □

This property cannot be extended directly to any SIfBm for which  $H < 1/2$ , since we do not have  $\mathbf{E}[|\Delta \mathbf{B}_C^H|^2] = m(C)^{2H}$  for all  $C \in \mathcal{C}$  (see [23]). However, the results of Proposition 6.2 hold in the specific case of rectangles of  $\mathbf{R}^N$ , i.e. for the multiparameter fractional Brownian motion (see Remark 6.9).

**6.2. Proof of the uniform a.s. pointwise exponent of the SIfBm.** In [1], the isotropic fractional Brownian field is proved to have a uniform pointwise exponent equal to  $H$  using techniques such as local times; and in [11], the same result holds for the regular multifractional Brownian motion, with a proof based on the integral representation of the mBm. This result relies on tools that are not available in the set-indexed framework, although some attempts have been made to introduce set-indexed local times ([26]).

In [11], the following technical lemma is proved for a multifractional Brownian motion. We restrict it to fBm's case:

**Lemma 6.3.** *Let  $B^H = \{B_t^H, t \in \mathbf{R}_+\}$  be a fractional Brownian motion of index  $H \in (0, 1)$ . Let  $\epsilon > 0$ ,  $\rho > 0$ ,  $0 \leq s < t$ ,  $n \in \mathbf{N}^*$  and  $\delta u = \frac{\rho}{n}$ . Then, let  $u_0 = s$  and for all  $k \in \{0, \dots, n\}$ ,  $u_{k+1} = u_k + \delta u$ . We have the following:*

$$\mathbf{P} \left( \bigcap_{k=1}^n \{|B_{u_k}^H - B_{u_{k-1}}^H| < \rho^{H+\epsilon}\} \right) \leq \left( \frac{2}{\sqrt{2\pi}} \right)^n \left( \frac{\rho^{H+\epsilon}}{C \cdot (\delta u)^H} \right)^n,$$

where  $C$  is a constant depending only on  $H$ .

In the sequel, for  $U \subset V \in \mathcal{A}$ , we denote by  $\mathcal{R}(f, U \rightarrow V)$ , the range of the elementary flow  $f : [0, d] \rightarrow \mathcal{A}$  such that  $f(0) = U$  and  $f(d) = V$ , where  $d = d_m(U, V)$  (the distance considered here is always  $d_m = m(\bullet \triangle \bullet)$ ). Hence  $\mathcal{R}(f, U \rightarrow V)$  is a totally ordered subset of  $\mathcal{A}$  which forms a continuum. We also denote by  $\mathcal{R}_n(f, U)$ , the range  $\mathcal{R}(f, U \rightarrow g_n(U))$ . Since the choice of a particular  $f$  does not matter, these notations can be used without specifying  $f$ , considering that a choice has been made.

**Lemma 6.4.** *Let  $\mathbf{B}^H$  be a SIIfBm on  $(\mathcal{A}, \mathcal{T}, m)$  of index  $H \in (0, \frac{1}{2}]$ . Let  $U \in \mathcal{A}$ ,  $i \in \mathbf{N}$  and  $\rho_i = d_m(U, g_i(U))$ . Let  $\epsilon > 0$ ,  $n \in \mathbf{N}^*$ . In any  $\mathcal{R}_i(f, U)$ , there exist an increasing sequence  $(U_j)_{0 \leq j \leq n}$  such that  $U_0 = U$ ,  $U_n = g_i(U)$ , and  $\delta U = d_m(U_{j-1}, U_j) = \frac{\rho_i}{n}$  for all  $j \in \llbracket 1, n \rrbracket$ . Then,*

$$\mathbf{P} \left( \bigcap_{k=1}^n \{|\mathbf{B}_{U_k}^H - \mathbf{B}_{U_{k-1}}^H| < \rho_i^{H+\epsilon}\} \right) \leq \left( \frac{2}{\sqrt{2\pi}} \right)^n \left( \frac{\rho_i^{H+\epsilon}}{\sigma} \right)^n$$

where  $\sigma = C \cdot (\delta U)^H$  and  $C > 0$  only depends on  $H$ . Equivalently, there exists a constant  $\tilde{C} > 0$ , which only depends on  $H$ , such that

$$\mathbf{P} \left( \bigcap_{k=1}^n \{|\mathbf{B}_{U_k}^H - \mathbf{B}_{U_{k-1}}^H| < \rho_i^{H+\epsilon}\} \right) \leq \left( \tilde{C} n^H \rho_i^\epsilon \right)^n. \quad (6.3)$$

*Proof.* Let us consider the range  $\mathcal{R}_i(f, U)$  of a flow  $f$  connecting  $U$  to  $g_i(U)$ . The standard projection of  $X = \mathbf{B}^H$  on  $f$  is a standard fractional Brownian motion that we denote  $X^{f,m} = \{X_t^{f,m}, t \in [0, \rho_i]\}$ . As usual,  $\theta = m \circ f$  and in the present situation,  $\theta : [0, \rho_i] \rightarrow [m(U), m(g_i(U))]$ . For  $k \in \llbracket 0, n \rrbracket$ , let  $u_k = m(U) + k \cdot \frac{\rho_i}{n}$  and define  $U_k = f \circ \theta^{-1}(u_k)$ . The  $U_k$ 's constitute the sequence of the statement and we remark that

$$\mathbf{P} \left( \bigcap_{k=1}^p \{|X_{U_k} - X_{U_{k-1}}| < \rho_n^{H+\epsilon}\} \right) = \mathbf{P} \left( \bigcap_{k=1}^p \{|X_{u_k}^{f,m} - X_{u_{k-1}}^{f,m}| < \rho_n^{H+\epsilon}\} \right).$$

The result follows from Lemma 6.3.  $\square$

The following Proposition 6.5 is the key result to prove the uniform almost sure upper bound for the SIIfBm.

**Proposition 6.5.** *Let  $\mathbf{B}^H$  be a SIIfBm on  $(\mathcal{A}, \mathcal{T}, m)$  of parameter  $H \in (0, 1/2]$ . We assume that  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  endowed with the distance  $d_m$  satisfies Assumption 1 and that Condition (6.2) holds.*

*Then, with probability one, for all  $\epsilon > 0$ , there exists a random variable  $h > 0$  such*

that for all  $\rho \leq h(\omega)$  and for all  $U_0 \in \mathcal{A}$ ,

$$\sup_{U, V \in B_{d_{\mathcal{A}}}(U_0, \rho)} |\mathbf{B}_U^H - \mathbf{B}_V^H| \geq \rho^{H+\epsilon}.$$

*Proof.* Let us fix  $\epsilon > 0$ . For all  $U \in \mathcal{A}$ , let  $\rho_{n,U} = d_m(U, g_n(U))$  and  $p_{n,U} = \lfloor \rho_{n,U}^{-\epsilon} \rfloor$ . For all  $N \in \mathbf{N}^*$ , we consider the event

$$A_N = \bigcup_{n \geq N} \bigcup_{U \in \mathcal{A}_n} \{ \forall V, W \in \mathcal{R}_n(f, U), |X_V - X_W| < \rho_{n,U}^{H+\epsilon} \}.$$

We have

$$\begin{aligned} \mathbf{P}(A_N) &\leq \sum_{n \geq N} \sum_{U \in \mathcal{A}_n} \mathbf{P}(\forall V, W \in \mathcal{R}_n(f, U), |X_V - X_W| < \rho_{n,U}^{H+\epsilon}) \\ &\leq \sum_{n \geq N} \sum_{U \in \mathcal{A}_n} \mathbf{P}\left(\bigcap_{k=1}^{p_{n,U}} \{|X_{U_k} - X_{U_{k-1}}| < \rho_{n,U}^{H+\epsilon}\}\right), \end{aligned}$$

where  $U_0, \dots, U_{p_{n,U}}$  are defined as in Lemma 6.4.

Following equation (6.3) and since  $\rho_{n,U} = d_{\mathcal{A}}(U, g_n(U)) \leq M_1 k_n^{-1/q_{\mathcal{A}}}$ , there exist positive constants  $C_1$  and  $C_2$  such that

$$\begin{aligned} \mathbf{P}\left(\bigcap_{k=1}^{p_{n,U}} \{|X_{U_k} - X_{U_{k-1}}| < \rho_{n,U}^{H+\epsilon}\}\right) &\leq \left(C_1 \rho_{n,U}^{\epsilon(1-H)}\right)^{\rho_{n,U}^{-\epsilon}} \\ &\leq \left(C_2 k_n^{-1/q_{\mathcal{A}}}\right)^{\epsilon(1-H)(M_1^{-\epsilon} k_n^{\epsilon/q_{\mathcal{A}}} - 1)}. \end{aligned}$$

Getting back to the previous equation, we obtain

$$\mathbf{P}(A_N) \leq \sum_{n \geq N} k_n \left(C_2 k_n^{-1/q_{\mathcal{A}}}\right)^{\epsilon(1-H)(M_1^{-\epsilon} k_n^{\epsilon/q_{\mathcal{A}}} - 1)} = R_N.$$

Since  $k_n$  grows faster than  $n$ , we can easily show that  $\sum_{N \in \mathbf{N}^*} R_N < \infty$ . Hence, Borel-Cantelli Lemma implies the existence of a random variable  $N(\omega)$  such that: with probability one, for all  $n \geq N(\omega)$  and for all  $U \in \mathcal{A}_n$ ,

$$\exists V, W \in \mathcal{R}_n(f, U); \quad |X_V - X_W| \geq \rho_{n,U}^{H+\epsilon}. \quad (6.4)$$

For  $U_0 \in \mathcal{A}$  and  $\rho > 0$ , Assumption (6.2) gives the existence of  $\mathcal{R}_n(f, U) \subset B_{d_{\mathcal{A}}}(U_0, \rho)$ , for some  $n \geq N(\omega)$  and  $U \in \mathcal{A}$  such that  $\rho_{n,U} \geq \eta\rho$ . Then, there exist  $V, W \in \mathcal{A}$  (the same that in (6.4)), such that

$$|X_V - X_W| \geq \rho_{n,U}^{H+\epsilon} \geq (\eta^{H+\epsilon}) \rho^{H+\epsilon}$$

which concludes the proof.  $\square$

As a consequence of Proposition 6.5, with probability one, the random pointwise Hölder exponent of a SifBm is uniformly lower than  $H$  (and thus, equal to  $H$ , by Theorem 5.4), provided that Assumptions 1, 2 and the additional requirement (6.2) hold.

**6.3. Hölder exponents of the SIOU process.** Theorems 5.3 and 5.4 can be also applied to derive Hölder exponents of the set-indexed Ornstein-Uhlenbeck (SIOU in short) process, studied in [10]. This process was introduced as an example of set-indexed process satisfying some stationarity and Markov properties.

A mean-zero Gaussian process  $Y = \{Y_U; U \in \mathcal{A}\}$ , where  $\mathcal{A}$  is an indexing collection on the measure space  $(\mathcal{T}, m)$ , is called a *stationary set-indexed Ornstein-Uhlenbeck process* if

$$\forall U, V \in \mathcal{A}, \quad \mathbf{E}[Y_U Y_V] = \frac{\sigma^2}{2\gamma} \exp(-\gamma m(U \triangle V)),$$

for given positive constants  $\gamma$  and  $\sigma$ .

Fixing  $U_0 \in \mathcal{A}$ , and for all  $U, V$  close to  $U_0$  for the metric  $d_m$ ,  $\mathbf{E}[|Y_U - Y_V|^2] = \frac{\sigma^2}{\gamma}(1 - e^{-\gamma m(U \triangle V)})$  implies that  $\mathbf{E}[|Y_U - Y_V|^2] = \sigma^2 [m(U \triangle V) + o(m(U \triangle V))]$ . This leads to  $\alpha_Y(U_0) = \tilde{\alpha}_Y(U_0) = 1/2$ . Consequently, the following result follows directly from Theorem 5.4.

**Theorem 6.6.** *Let  $Y = \{Y_U; U \in \mathcal{A}\}$  be a stationary set-indexed Ornstein-Uhlenbeck process on  $(\mathcal{T}, \mathcal{A}, m)$ . Assume that the subclasses  $(\mathcal{A}_n)_{n \in \mathbf{N}}$  of  $\mathcal{A}$  satisfy Assumptions 1 and 2.*

*Then, the pointwise and local Hölder exponents satisfy, with probability one,*

$$\forall U_0 \in \mathcal{A}, \quad \tilde{\alpha}_Y(U_0) = \frac{1}{2} \quad \text{and} \quad \alpha_Y(U_0) \geq \frac{1}{2}$$

*and  $\forall U_0 \in \mathcal{A}$ ,  $\alpha_Y(U_0) = \frac{1}{2}$  a.s.*

*As another consequence of the previous remark, the equality holds for the  $\mathcal{C}$ -Hölder exponents, for all  $U_0 \in \mathcal{A}$ , almost surely.*

As mentioned in the case of the SIFBm, the computation of the exponent of pointwise continuity requires a fine estimation of the variance of the process over  $\mathcal{C}$ . When  $\mathcal{A}$  is the collection of the rectangles of  $\mathbf{R}_+^N$ , the estimation of  $\mathbf{E}[|\Delta Y_{C_n(t)}|^2]$  is easier, as the example of the SIOU process shows.

**Lemma 6.7.** *Let  $\mathcal{A} = \{[0, t] : t \in [0, 1]^N\}$  endowed with the usual dissecting class  $(\mathcal{A}_n)$  made of the dyadics. Let  $t \in (0, 1)^N$ ,  $t = (t_1, \dots, t_N)$  and define:*

$$t_j^n = \begin{cases} t_j & \text{if } 2^n t_j \in \mathbf{N} \\ 2^{-n} \lfloor 2^n t_j + 1 \rfloor & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{t}_k^n = \begin{cases} 2^{-n} \lfloor 2^n t_k - 1 \rfloor & \text{if } 2^n t_k \in \mathbf{N} \\ 2^{-n} \lfloor 2^n t_k \rfloor & \text{otherwise.} \end{cases}$$

*Then,*

$$C_n(t) = [0, (t_1^n, \dots, t_N^n)] \setminus \bigcup_{k=1}^N [0, (t_1^n, \dots, \tilde{t}_k^n, \dots, t_N^n)].$$

*Proof.* We recall that  $C_n(t)$ , the left-neighbourhood of  $A_t$  in  $\mathcal{A}_n$ , is defined as  $\bigcap_{\substack{C \in \mathcal{C}_n \\ t \in C}} C$ . In the particular case of the rectangles, it corresponds to the expression given in the lemma.  $\square$

As usual, let  $\lambda$  be the Lebesgue measure of  $\mathbf{R}^N$ . A direct consequence of this result is that any Gaussian process  $X$  satisfying the assumptions of Corollary 2.10 satisfies,



for all  $t \in [0, 1]^N$  and for all  $\omega$ ,

$$\tilde{\alpha}_{X,c}(A_t) \leq \alpha_{X,c}(A_t) \leq \alpha_X^{pc}(t),$$

with respect to the Lebesgue measure  $\lambda$  and the distance  $d_\lambda$ .

More precise results are available for the SIOU process and the SIfBm.

**Proposition 6.8.** *Let  $Y = \{Y_U : U \in \mathcal{A}\}$  be a SIOU process, where  $\mathcal{A}$  refers to the rectangles of  $[0, 1]^N$  as in the Lemma 6.7. Then, the pointwise continuity of  $Y$  with respect to the Lebesgue measure  $\lambda$  of  $\mathbf{R}^N$  satisfies*

$$\forall t_0 \in [0, 1]^N, \quad \mathbf{P} \left( \alpha_Y^{pc}(t_0) = \alpha_Y^{pc}(t_0) = \frac{1}{2} \right) = 1,$$

and,

$$\mathbf{P} \left( \forall t_0 \in [0, 1]^N, \quad \alpha_Y^{pc}(t_0) \geq \alpha_Y^{pc}(t_0) = \frac{1}{2} \right) = 1.$$

*Proof.* For sake of readability, the proof is written for  $N = 2$ . Let  $t = (t_1, t_2) \in [0, 1]^N$ . To show there is no difference in the final result, we assume  $t_1$  is dyadic and  $t_2$  is not. Let  $k, l \in \mathbf{N}, k < 2^l$  such that  $t_1 = k \cdot 2^{-l}$ . Let  $n \in \mathbf{N}, n \geq l$ .

First, we notice that, by Lemma 6.7,

$$C_n(t) = [0, (t_1, 2^{-n} \lfloor 2^n t_2 + 1 \rfloor)] \setminus \{ [0, (2^{-n} \lfloor 2^n t_1 - 1 \rfloor, 2^{-n} \lfloor 2^n t_2 + 1 \rfloor)] \cup [0, (t_1, 2^{-n} \lfloor 2^n t_2 \rfloor)] \}.$$

Re-writing this for short  $C_n(t) = A_n \setminus \{B_{1,n} \cup B_{2,n}\}$ , the inclusion-exclusion formula gives

$$\begin{aligned} \mathbf{E}[|\Delta Y_{C_n(t)}|^2] &= \mathbf{E}[Y_{A_n}^2 + Y_{B_{1,n}}^2 + Y_{B_{2,n}}^2 + Y_{B_{1,n} \cap B_{2,n}}^2 - 2Y_{A_n}Y_{B_{1,n}} - 2Y_{A_n}Y_{B_{2,n}} \\ &\quad + 2Y_{A_n}Y_{B_{1,n} \cap B_{2,n}} + 2Y_{B_{1,n}}Y_{B_{2,n}} - 2Y_{B_{1,n}}Y_{B_{1,n} \cap B_{2,n}} - 2Y_{B_{2,n}}Y_{B_{1,n} \cap B_{2,n}}]. \end{aligned}$$

Combined with the covariance of the SIOU, a second-order Taylor expansion gives:

$$\mathbf{E}[|\Delta Y_{C_n(t)}|^2] = \frac{\sigma^2}{2\gamma} (8\gamma \cdot 2^{-2n} + 16\gamma^2 \cdot 2^{-4n} \lfloor 2^n t_1 \rfloor \cdot \lfloor 2^n t_2 \rfloor + o(2^{-2n})).$$

Considering the fact that  $\lambda(C_n(t)) = 2^{-2n}$ , the previous expansion implies  $\alpha_Y^{pc}(t) = \frac{1}{2}$ . Therefore, Corollary 5.8 gives the result.  $\square$

**Remark 6.9.** *With the notations of Proposition 6.8, we can consider the case of the SIfBm  $\mathbf{B}^H$  indexed by  $\mathcal{A} = \{[0, t], t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$ ,*

$$\begin{aligned} \mathbf{E} [|\Delta \mathbf{B}_{C_n(t)}^H|^2] &= m(A_n \setminus B_{1,n})^{2H} + m(A_n \setminus B_{2,n})^{2H} - m(A_n \setminus (B_{1,n} \cap B_{2,n}))^{2H} \\ &\quad - m(B_{1,n} \triangle B_{2,n})^{2H} + m(B_{1,n} \setminus B_{2,n})^{2H} + m(B_{2,n} \setminus B_{1,n})^{2H}. \end{aligned}$$

*Then, the same development as the previous proof gives  $\alpha_{\mathbf{B}^H}^{pc}(t_0) = H$  for all  $t_0 \in [0, 1]^N$ . Consequently, we can state:*

$$\forall t_0 \in [0, 1]^N, \quad \mathbf{P} (\alpha_{\mathbf{B}^H}^{pc}(t_0) = \alpha_{\mathbf{B}^H}^{pc}(t_0) = H) = 1,$$

and,

$$\mathbf{P} (\forall t_0 \in [0, 1]^N, \quad \alpha_{\mathbf{B}^H}^{pc}(t_0) \geq \alpha_{\mathbf{B}^H}^{pc}(t_0) = H) = 1.$$

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